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Chains of pseudocompact group topologies[1]

Dikran Dikranjan*

Dipartimento di Matematica e Informatica, Università degli Studi di Udine, via delle Scienze 206 (località Rizzi), 33100 Udine, Italy

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Abstract

We show that for several large classes of groups G, the poset $\mathscr{P}_\sigma(G)$ of all pseudocompact group topologies of weight σ on G contains a poset isomorphic to the power set of σ whenever $\mathscr{P}_\sigma(G) \neq \emptyset$. This permits to describe in purely set-theoretic terms when G has bounded (from above) chains of pseudocompact group topologies of weight σ. Moreover, we show that these topologies may additionally have some of the following properties: linear (in particular, zero-dimensional), connected and locally connected, disconnected and locally connected, connected and nonlocally connected, etc. It turns out that the question of whether the cardinality of unbounded chains of pseudocompact group topologies of weight ω_1 on $\mathbb{R}$ is larger than that of bounded ones cannot be answered in ZFC. We answer also questions posed by Comfort and Remus (1994) on pseudocompact topologization of a given weight. © 1998 Elsevier Science B.V.

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0. Introduction

A topological group G is *precompact* if it is (algebraically and topologically isomorphic to) a subgroup of a compact group, or equivalently, if the two-sided uniformity completion $\hat{G}$ of G is a compact group [45]. A topological space is *pseudocompact* if every real-valued continuous function defined on it is bounded [26]. Let $\mathscr{P}(G)$ (resp. $\mathscr{B}(G)$) denote the ordered set of all pseudocompact (resp. precompact) group topologies on a group G ordered by inclusion. Since pseudocompact groups are precompact $\mathscr{P}(G) \subseteq \mathscr{B}(G)$ [15]. For every cardinal σ denote by $\mathscr{B}_\sigma(G)$ the subset of $\mathscr{B}(G)$ consisting of topologies of weight σ and let $\mathscr{P}_\sigma(G) = \mathscr{P}(G) \cap \mathscr{B}_\sigma(G)$.

* E-mail: dikranja@dimi.uniud.it.

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Comfort and Ross [15] showed that $\mathscr{B}(G) \neq \emptyset$ for each infinite Abelian group G and actually even $|\mathscr{B}(G)| = 2^{2^{|G|}}$ — the highest amount possible (see [4, 35]). There are non-Abelian groups G with $\mathscr{B}(G) = \emptyset$: von Neumann and Wigner [43] showed this for the group $\mathrm{SL}(2,\mathbb{C})$ of all complex 2×2 matrices having determinant 1 (see also [6, 9.8]), and Gaughan [28] established the same for the group $S(X)$ of permutations of an infinite set X (another example is given in Section 3, Remark 3.5). By another classical result of van der Waerden $|\mathscr{B}(G)| = |\mathscr{B}_\omega(G)| = 1$ for a compact connected semisimple Lie group G, i.e. these groups admit a unique precompact topology, namely the given compact metrizable group topology.

A detailed study of the poset $\mathscr{B}(G)$ and its poset invariants as height, depth and width (i.e. maximum size of well-ordered, anti-well-ordered subsets and antichains, respectively) in the case when G is Abelian was carried out in [4]. These results were extended later by Remus [35] for groups admitting "large" Abelian quotients (i.e. satisfying $|G/G'| = |G|$), in particular for free groups. Long chains in $\mathscr{B}(G)$ were produced by Comfort and Remus in [8]. They found also long chains of group topologies from other classes of group topologies: nonprecompact, metrizable and linear, as well as nonpseudocompact linear topologies on free groups. (A topological group G is said to have *linear topology* if the open normal subgroups of G form a base of open neighbourhoods of 1.)

For cardinals λ and σ let $C(\sigma,\lambda)$ stay for the set-theoretic assumption "there is a chain of length λ in the power set $\mathbf{P}(\sigma)$ of σ" (see [8], for other equivalent condition see [2]). It was shown in [8] that in case the group G is Abelian or free the existence of a chain of length λ in $\mathscr{B}(G)$ is equivalent to $C(2^{|G|},\lambda)$, so in particular does not depend on the algebraic properties of G but only on $|G|$. A generalization of this result for groups G satisfying $|G/G'| = |G|$ was announced without proof in [10, Theorem II] (see also [7, Section 3.10.I]).

The case of pseudocompact group topologies is substantially harder. The first restraint on the algebraic structure of infinite pseudocompact groups was found by van Douwen [42]. Namely, he showed that the cardinality of such a group should be at least $\mathfrak{c}$ (the cardinality of continuum) and $|G|$ cannot be a strong limit cardinal of countable cofinality. There was recently a substantial progress in the study of the class $\mathfrak{P}$ of groups admitting pseudocompact group topologies [7, Section 3.10.B; 9; 11; 12; 20–23; 38; 40]. This stimulated us to provide in the present paper various classes of groups (see (i)–(iv) below) with chains of pseudocompact group topologies of a given weight σ and show that these chains have the maximal possible length.

Following [7, Notation 3.10.1(a)] we denote by $\mathfrak{P}(\sigma)$ the class of abstract groups admitting pseudocompact group topology of weight σ, i.e. with $\mathscr{P}_\sigma(G) \neq \emptyset$. By [13], $G \in \mathfrak{P}(\sigma)$ yields $\delta(\sigma) \leq |G| \leq 2^\sigma$, where $\delta(\sigma)$ is the G_δ-density of $\{0,1\}^\sigma$ (see Section 1 for an equivalent definition in set-theoretic terms). We shall denote this condition by $\mathrm{Ps}(|G|,\sigma)$. In Section 1 we provide all necessary facts concerning the condition $\mathrm{Ps}(\tau,\sigma)$ between two cardinals τ and σ. It turns out that $G \in \mathfrak{P}(\sigma)$ can be described by means of this condition involving various cardinal invariants of the group

G, such as its cardinality, free rank $r(G)$, the p-ranks $r_p(G)$, Ulm–Kaplansky invariants of G, etc. [23].

Our main result says: *if* $G \in \mathfrak{P}(\sigma)$ *then the poset* $\mathscr{P}_\sigma(G)$ *contains a copy of the Boolean algebra* $\mathbf{P}(\sigma)$ *provided the group* G *has one of the properties*: (i) *relatively free*, (ii) *torsion Abelian*, (iii) *Abelian with* $\delta(\log|G|) \leq r(G)$ (*in particular*, $|G| = r(G)$, *or even, torsion-free Abelian*); (iv) *divisible Abelian* (Theorems 2.3, 4.1, 5.1). Condition (iii) means that G admits connected pseudocompact group topologies of weight σ (see Theorem 5.1). Of course, this yields a lot of lower bounds concerning the size of various poset-invariants of $\mathscr{P}_\sigma(G)$, maximum length of chains, width, height, depth, etc. For example, for a group $G \in \mathfrak{P}(\sigma)$ of the above type $\mathscr{P}_\sigma(G)$ has chains of length λ if $C(\sigma, \lambda)$ holds (so that $\mathscr{P}_\sigma(G)$ has chains of length σ^+). Obviously, in such a case $\mathscr{P}_\sigma(G)$ (as $\mathbf{P}(\sigma)$) has also antichains of size 2^σ. In this paper we shall consider only the chains.

For a set X denote by $\mathbf{P}_\sigma(X)$ the family of all subsets of X of cardinality σ. It follows easily from Tanaka–Krein duality for compact groups that for an arbitrary maximally almost periodic group G the poset $\mathscr{B}_\sigma(G)$ embeds into $\mathbf{P}_\sigma(X)$ (where X is the Tanaka dual of the Bohr compactification of G, see Fact 1.12). Hence every chain with top element (= *bounded chain*) in $\mathscr{B}_\sigma(G)$ gives rise to a chain with the same size in $\mathbf{P}(\sigma)$. Therefore the existence of a bounded chain of size λ in $\mathscr{B}_\sigma(G)$ implies $C(\sigma, \lambda)$. Conversely, our main result implies that in all cases (i)–(iv) mentioned above $C(\sigma, \lambda)$ implies the existence of a bounded chain of size λ in $\mathscr{P}_\sigma(G)$, and hence $\mathscr{B}_\sigma(G)$. This resolves completely the question in the case of bounded chains.

In Section 6 we give all results concerning unbounded chains and we compare lengths of bounded and unbounded chains. Here we lean essentially on a general approach to precompact topologization developed in a forthcoming paper [3] and anticipated in [16, 17]: for a group G close to being Abelian the poset $\mathscr{B}_\sigma(G)$ is quasi-isomorphic to the poset $\mathbf{P}_\sigma(2^{|G|})$ (see Theorem 6.1). This permits an easy calculation of the poset cardinal invariants of $\mathscr{B}_\sigma(G)$ and yields further information on the poset structure of $\mathscr{P}_\sigma(G)$: for example the existence of chains of length λ in $\mathscr{B}_\sigma(G)$ implies the existence of chains of length λ in the poset $\mathbf{P}_\sigma(\sigma^+)$. Under some additional set-theoretical assumptions (for example, the generalized continuum hypothesis, briefly GCH) this may yield $C(\sigma, \lambda)$, so that unbounded chains "have the same length" as bounded ones (Theorem 6.3). In Theorem 6.4 we provide a sufficient condition for having the same chain lengths in $\mathscr{P}_\sigma(G)$ and $\mathscr{B}_\sigma(G)$ (this holds also the under GCH, Theorem 6.3) and discuss its necessity under the singular cardinal hypothesis (SCH) in Remark 6.8. In Theorem 6.9 we prove that ZFC cannot decide whether bounded and unbounded chains in $\mathscr{P}_{\omega_1}(\mathbb{R})$ have the same length (its counterpart in the precompact case for $\mathscr{B}_{\omega_1}(\mathbb{Z})$, was proved in [3]).

In this paper we also answer several questions from [11, 12]. As already noted above, $G \in \mathfrak{P}(\sigma)$ (i.e., $\mathscr{P}_\sigma(G) \neq \emptyset$) yields $\mathrm{Ps}(|G|, \sigma)$. The question of whether the converse is also true was raised in [12, Question 3.7]. Call a group $G \in \mathfrak{P}$ a CR-*group* if $\mathrm{Ps}(|G|, \sigma)$ always implies $\mathscr{P}_\sigma(G) \neq \emptyset$. In these terms Question 3.7 from [12] sounds as follows:

0.1. Question. (a) Is every group $G \in \mathfrak{P}$ necessarily a CR-group?
(b) Does a group $G \in \mathfrak{P}$ satisfy also $G \in \mathfrak{P}(\log|G|)$?

Every CR-group G satisfies $G \in \mathfrak{P}(\log|G|)$, since $G \in \mathfrak{P}$ yields $\mathrm{Ps}(|G|, \log|G|)$ [12, Lemma 3.4]. We show below (Corollary 4.6) that the divisible Abelian groups of cardinality $> \mathfrak{c}$ have always the property (b) although they are not CR-groups in general (see Example 4.8 which answers negatively item (a) of Question 0.1 and we characterize the divisible Abelian CR-groups under GCH (Theorem 5.9). Since metrizable pseudocompact groups are compact, obviously a group $G \in \mathfrak{P}$ with $|G| = \mathfrak{c}$ satisfies (b) iff G admits a compact metrizable topology. If G is also Abelian, then G is a CR-group (Theorem 4.7). Now $\mathscr{P}_\omega(G)$ need not contain $\mathbf{P}(\omega)$, actually $|\mathscr{P}_\omega(G)| = 1$ may happen (see Remark 5.2(c)). We show that among the groups $G \in \mathfrak{P}$ with $|G| > \mathfrak{c}$ there are several classes which consist of CR-groups: the relatively free groups, torsion Abelian groups, torsion-free Abelian groups (more generally, Abelian groups G with $r(G) = |G|$).

1. Preliminaries

1.1. Cardinal functions related to pseudocompact topologization

The following purely set-theoretic condition $\mathrm{Ps}(\tau, \sigma)$ between infinite cardinals τ and σ was introduced in 1978 by Cater et al. [5]: the set $\{0,1\}^\sigma$ of all functions from (a set of cardinality) σ to the two-point set $\{0,1\}$ contains a subset F of cardinality τ whose projection on every countable subproduct $\{0,1\}^A$ is a surjection. The notation $\mathrm{Ps}(\tau, \sigma)$ was adopted for the first time in [20, 21] (for a justification see Fact 1.1 below). In case $\mathrm{Ps}(\tau, \sigma)$ holds for infinite cardinals τ and σ we say that τ is *σ-admissible* and for a cardinal $\sigma \geq \omega$ we set $\delta(\sigma) = \min\{\tau : \mathrm{Ps}(\tau, \sigma) \text{ holds}\}$ following [5]. Obviously $\mathrm{Ps}(\tau, \sigma)$ is equivalent to $\delta(\sigma) \leq \tau \leq 2^\sigma$. An infinite cardinal τ is *admissible* if it is σ-admissible for some σ, in such a case we set $\Pi(\tau) = \sup\{\sigma : \mathrm{Ps}(\tau, \sigma) \text{ holds}\}$.

The following fundamental fact explains the choice of our notation $\mathrm{Ps}(\tau, \sigma)$.

1.1 Fact (Comfort and Robertson [13]). *If $G \in \mathfrak{P}(\sigma)$ then* $\mathrm{Ps}(|G|, \sigma)$ *holds; viceversa, if* $\mathrm{Ps}(\tau, \sigma)$ *holds, then there exists $G \in \mathfrak{P}(\sigma)$ with $|G| = \tau$.*

This fact implies that a cardinal τ is admissible iff there exists a group $G \in \mathfrak{P}$ with $|G| = \tau$. To see other related notions we need some more definitions and notation.

The SCH is the set-theoretic assumption which ensures that $\tau^\lambda \leq 2^\lambda \cdot \tau^+$ for infinite cardinals τ and λ (so that an infinite cardinal $\tau \geq \mathfrak{c}$ having uncountable cofinality satisfies $\tau^\omega = \tau$; note that the converse is always true). It is known that GCH implies SCH, but the latter is much weaker [5, p. 310].

For an infinite cardinal τ set $\log \tau = \min\{\alpha : 2^\alpha \geq \tau\}$ and $2^{<\tau} = \sup\{2^\lambda : \lambda < \tau\}$. Note that $2^{<\tau}$ is a proper limit iff $\log 2^{<\tau} = \tau$, i.e., $2^\lambda < 2^{<\tau}$ whenever $\lambda < \tau$. The cardinal τ is a *strong limit* if $\lambda < \tau$ yields $2^\lambda < \tau$, or equivalently, $2^{<\tau} = \tau = \log\tau$. Van Douwen [42] showed that the cardinality τ of an infinite pseudocompact group should be at least $\mathfrak{c}$ and τ cannot be a strong limit cardinal of countable cofinality. We call a cardinal with this property a *van Douwen cardinal*. Clearly, admissible cardinals are van Douwen. The converse is true under SCH [42] (see also [23]). In Lemma 1.3 below we show where can be located the nonadmissible van Douwen cardinals when SCH fails.

Our next lemma recollects some basic results concerning $\mathrm{Ps}(\tau,\sigma)$ which will be needed for applications in the sequel.

1.2. Lemma. (a) *Let σ and τ be infinite cardinals.*

(a1) $\mathrm{Ps}(\tau,\sigma)$ *implies* $\mathfrak{c} \le \tau \le 2^\sigma$ *and* $\sigma \le 2^\tau$,

(a2) $\mathrm{cf}(\delta(\sigma)) > \omega$, $\mathfrak{c} \le \delta(\sigma) \le 2^\sigma$ *and* $\delta(\sigma) \ge \log\sigma$.

(a3) $\sigma \le \sigma'$ *implies* $\delta(\sigma) \le \delta(\sigma')$.

(a4) $\delta(\sigma) \le (\log\sigma)^\omega$.

(b) *If τ is admissible, then τ is* $\log\tau$*-admissible.*

(c) *Let for $i \in I$ σ_i and τ_i be infinite cardinals such that* $\mathrm{Ps}(\tau_i,\sigma_i)$ *holds for $i \in I$. Then also* $\mathrm{Ps}(\min\{\tau_i : i \in I\}, \min\{\sigma_i : i \in I\})$ *and* $\mathrm{Ps}(\prod\{\tau_i : i \in I\}, \sup\{\sigma_i : i \in I\})$ *hold. In particular, if I is finite, then* $\mathrm{Ps}(\max\{\tau_i : i \in I\}, \max\{\sigma_i : i \in I\})$ *holds.*

Proof. (a) is proved in [5, Lemmas 1.1–1.5, Theorem 1.5] and (b) is proved in [12, Lemma 3.4].

To check (c) set $\tau_{i'} = \min\{\tau_i : i \in I\}$ and $\sigma_{i''} = \min\{\sigma_i : i \in I\}$. If $i' = i''$ there is nothing to prove. Otherwise, $\sigma_{i'} \ge \sigma_{i''}$. Then $\delta(\sigma_{i''}) \le \delta(\sigma_{i'}) \le \tau_{i'} \le \tau_{i''} \le 2^{\sigma_{i''}}$ which proves $\mathrm{Ps}(\tau_{i'},\sigma_{i''})$. To prove the second part note that according to Fact 1.1, for each $i \in I$ there exists a pseudocompact group G_i of weight σ_i and cardinality τ_i. Then by Comfort–Ross' theorem [15] $\prod G_i$ is a pseudocompact group of weight $\sigma = \sup\{\sigma_i : i \in I\}$, thus $\mathrm{Ps}(\prod\{\tau_i : i \in I\}, \sigma)$ holds. □

Items (a2) and (a4) of the lemma (namely, $\mathrm{cf}(\delta(\sigma)) > \omega$, $\delta(\sigma) \ge \mathfrak{c}$ and $\log\sigma \le \delta(\sigma) \le (\log\sigma)^\omega$) imply that $\delta(\sigma) = (\log\sigma)^\omega$ under SCH. It is announced in [11, 12] that the set-theoretic assumption

(M) $\quad \delta(\sigma) = (\log\sigma)^\omega \quad$ for all $\sigma \ge \omega$

is proved to be strictly weaker than SCH by Masaveu [30]. As noted already in [5, Problem] (see also [13]), it is not known if (M) is a theorem in ZFC.

We propose now a cardinal function useful when working without the assumption of SCH. For $\tau \ge \omega$ set $\sqrt{\tau} = \min\{\alpha : \alpha^\omega > \tau\}$, i.e. now $\rho < \sqrt{\tau}$ for a cardinal ρ is an abbreviation of $\rho^\omega \le \tau$. Obviously, $\sqrt{\tau} \le \tau^+$; with $\sqrt{\tau} = \tau^+$ iff $\tau^\omega = \tau$. Note that $\log\tau \le \sqrt{\tau}$, equality holds iff $(\log\tau)^\omega > \tau$. Clearly for every $\tau \ge \mathfrak{c}$ SCH yields $\sqrt{\tau} = \tau$ iff $\tau^\omega \ne \tau$ iff $\mathrm{cf}(\tau) = \omega$. By (b) (M) yields that a cardinal τ is admissible iff $(\log\log\tau)^\omega \le \tau$ ([12, Theorem 3.8(b)]), i.e. $\log\log\tau < \sqrt{\tau}$. Let us isolate this weaker statements as:

(WM) every admissible τ satisfies $\log\log\tau < \sqrt{\tau}$.

We do not know if (WM) is strictly weaker than (M).

1.3. Lemma. (WM) *is equivalent to the following statement*:

(S) *For every strong limit cardinal* θ *with* $\mathrm{cf}(\theta) = \omega$ *the open (possibly empty) interval* $(\theta, \theta^{\omega})$ *contains no admissible cardinals.*

Proof. (S) $\to$ (WM) If $\log\log\tau = \sqrt{\tau}$ for an admissible cardinal τ, then also $\log\log\tau = \log\tau = \sqrt{\tau}$, so that $\log\tau$ is a strong limit. Moreover, $\log\tau = \sqrt{\tau}$ yields $\log\tau \neq (\log\tau)^{\omega}$, so that $\mathrm{cf}(\log\tau) = \omega$. Moreover, since τ is a van Douwen cardinal and $\mathrm{cf}(\log\tau) = \omega$, we must have $\log\tau < \tau < (\log\tau)^{\omega}$. This contradicts (S) with $\theta = \log\tau$.

(WM) $\to$ (S) For strong limit cardinal θ with $\mathrm{cf}(\theta) = \omega$ every cardinal τ in the open interval $(\theta, \theta^{\omega})$ satisfies $\theta = \log\tau = \log\log\tau = \sqrt{\tau} < \tau$, hence cannot be admissible according to (WM). $\square$

In the equivalent form (S) one can see how weaker is (WM) than SCH, where *all* open intervals $(\theta, \theta^{\omega})$, with $\theta \geq \mathfrak{c}$ and $\mathrm{cf}(\theta) = \omega$, are imposed to be empty. We do not know the answer to the following.

1.4. Question. Is (S) (or, equivalently, (WM)) a theorem in ZFC?

We note that by (a3) and (b) of Lemma 1.2 the cardinals σ's such that an admissible cardinal τ is σ-admissible form an interval with least element $\log\tau$. Our next aim will be to compute the upper bound $\Pi(\tau)$ of this interval. We give first a lemma to produce σ-admissible cardinals in ZFC.

1.5. Lemma. *If* $\sigma^{\omega} \leq \tau \leq 2^{\sigma}$ *(i.e.* $\sigma \in [\log\tau, \sqrt{\tau})$*) for some* $\sigma \geq \omega$*, then* τ *is* κ*-admissible for each cardinal* κ *satisfying* $\log\tau \leq \kappa \leq 2^{\sigma}$. *Under* (M) *the converse is also true, i.e. if* $\mathrm{Ps}(\tau,\kappa)$ *holds, then* $\log\tau \leq \kappa \leq 2^{\sigma}$ *for some* $\sigma \in [\log\tau, \sqrt{\tau})$.

Proof. It follows from (a4) of Lemma 1.2 that $\delta(2^{\sigma}) \leq (\log(2^{\sigma}))^{\omega} \leq \sigma^{\omega} \leq \tau$, which obviously implies $\mathrm{Ps}(\tau, 2^{\sigma})$. Since $\log\tau \leq \kappa \leq 2^{\sigma}$, it follows that $\mathrm{Ps}(\tau,\kappa)$ holds as well.

Assume $\mathrm{Ps}(\tau,\kappa)$ and (M) hold. Then take $\sigma = \tau$ in case $\tau^{\omega} = \tau$. If $\tau^{\omega} \neq \tau$, then $\delta(\kappa)^{\omega} = \delta(\kappa) \leq \tau$, where the first equality follows from (M). Consider two cases. If $\tau \leq 2^{\delta(\kappa)}$, $\sigma := \delta(\kappa)$ works, since $\kappa \leq 2^{\delta(\kappa)}$. If $2^{\delta(\kappa)} \leq \tau$, $\sigma := 2^{\delta(\kappa)}$ works, since $\kappa \leq 2^{\log(\kappa)} \leq 2^{\delta(\kappa)} = \sigma$, so that $\sigma = \sigma^{\omega} \leq \tau \leq 2^{\kappa} \leq 2^{\sigma}$. $\square$

As an easy corollary of the above lemma one can see easily that if $2^{\alpha} \leq \tau \leq 2^{2^{\alpha}}$ for some $\alpha \geq \omega$, then τ is κ-admissible for each cardinal κ satisfying $\log\tau \leq \kappa \leq 2^{2^{\alpha}}$. In fact, take in the above lemma 2^{α} instead of σ to get $\mathrm{Ps}(\tau, 2^{2^{\alpha}})$ (this can be done since $(2^{\alpha})^{\omega} = 2^{\alpha}$).

1.6. Lemma. *Let τ be a cardinal.*

(a) *If τ is admissible and* $\mathrm{cf}(\tau) = \omega$, *then* $\Pi(\tau) \le 2^{<\tau}$. *If* $\log 2^{<\tau} = \tau$, *then* $\mathrm{Ps}(\tau, 2^{<\tau})$ *fails.*

(b) *If* $\log\log\tau < \sqrt{\tau}$ *(i.e.* $(\log\log\tau)^{\omega} \le \tau$), *then τ is admissible and* $\Pi(\tau) \ge 2^{<\sqrt{\tau}} \ge \tau$.

Proof. (a) Suppose $\mathrm{Ps}(\tau,\sigma)$ holds. Then $\delta(\sigma) \le \tau$. Our assumption on τ and (a2) give $\delta(\sigma) < \tau$, so that $\sigma \le 2^{\log\sigma} \le 2^{\delta(\sigma)} \le 2^{<\tau}$. Clearly, $\mathrm{Ps}(\tau, 2^{<\tau})$ fails if $2^{<\tau}$ is a proper limit.

(b) Assume $\log\log\tau < \sqrt{\tau}$. To prove that τ is admissible and $\Pi(\tau) \ge 2^{<\sqrt{\tau}}$ it suffices to show (by item (a2) of Lemma 1.2) that $\mathrm{Ps}(\tau, 2^{\rho})$ holds for all sufficiently big cardinals ρ such that $\log\log\tau \le \rho < \sqrt{\tau}$. Since $\rho^{\omega} \le \tau$ is ensured by our assumption $\rho < \sqrt{\tau}$, in order to be able to apply the above lemma, we have to ensure that also $\tau \le 2^{\rho}$ holds in such a case. Consider two cases.

Suppose first that $\log\tau = \sqrt{\tau}$ and pick a cardinal ρ such that $\log\log\tau \le \rho < \sqrt{\tau}$. Then our assumption entails $(\log\tau)^{\omega} > \tau$, thus $2^{\rho} \ge 2^{\log\log\tau} = (2^{\log\log\tau})^{\omega} \ge (\log\tau)^{\omega} \ge \tau \ge \rho^{\omega}$.

Suppose now that $\log\tau < \sqrt{\tau}$ and pick a cardinal ρ with $\log\tau \le \rho < \sqrt{\tau}$. Then $2^{\rho} \ge 2^{\log\tau} \ge \tau \ge \rho^{\omega}$.

To finish the proof of (b) note that our argument gives also $2^{<\sqrt{\tau}} \ge \tau$ since we have proved that $\tau \le 2^{\rho}$ for all sufficiently big cardinals ρ with $\rho < \sqrt{\tau}$. □

1.7. Corollary. *Assume* (WM). *Then* $\Pi(\tau) \ge 2^{<\sqrt{\tau}} \ge \tau$ *for every admissible cardinal τ.*

1.8. Question. Which of the following is a theorem in ZFC?

(a) $\Pi(\tau) \ge \tau$ for every admissible cardinal τ.

(b) $\Pi(\tau) > \log\tau$ for every admissible cardinal τ.

Note that (a) yields (b). Indeed, in case $\tau = \log\tau$ is a strong limit, van Douwen's theorem gives $\mathrm{cf}(\tau) > \omega$ which in this case yields $\tau = \tau^{\omega}$ with consequent $\mathrm{Ps}(\tau, 2^{\tau})$ and $\Pi(\tau) = 2^{\tau}$.

1.9. Proposition. *Assume* (M). *Then* $\Pi(\tau) = 2^{<\sqrt{\tau}}$ *holds for any admissible cardinal τ. Moreover,* $\mathrm{Ps}(\tau, 2^{<\sqrt{\tau}})$ *fails iff* $\log 2^{<\sqrt{\tau}} = \sqrt{\tau}$.

Proof. The inequality $\Pi(\tau) \ge 2^{<\sqrt{\tau}}$ follows from Corollary 1.7. To check the opposite inequality suppose $\mathrm{Ps}(\tau,\sigma)$ holds. Then $\delta(\sigma) = \delta(\sigma)^{\omega} \le \tau$, the equality being a consequence of (M). Hence $\delta(\sigma) < \sqrt{\tau}$. Then $\sigma \le 2^{\delta(\sigma)} \le 2^{<\sqrt{\tau}}$. This argument shows also that $\mathrm{Ps}(\tau, 2^{<\sqrt{\tau}})$ fails if $2^{<\sqrt{\tau}}$ is a proper limit. □

1.10. Corollary. *Assume SCH and let τ be an admissible cardinal.*

(a) *If* $\tau^\omega = \tau$ *then* $\mathrm{Ps}(\tau, 2^\tau)$ *holds, otherwise* $\Pi(\tau) = 2^{<\tau}$. *Moreover,* $\mathrm{Ps}(\tau, 2^{<\tau})$ *fails iff* $\log 2^{<\tau} = \tau$.

(b) $\mathrm{Ps}(\tau, 2^\tau)$ *fails precisely when* $\mathrm{cf}(\tau) = \omega$ *and* $\log 2^\tau = \tau$.

Proof. (a) If $\tau^\omega = \tau$ apply Lemma 1.5 with $\sigma = \tau$ to get $\mathrm{Ps}(\tau, 2^\tau)$. In case $\tau^\omega \neq \tau$ SCH implies $\sqrt{\tau} = \tau$. Now the above proposition applies.

(b) Assume $\mathrm{Ps}(\tau, 2^\tau)$ fails. Then $\mathrm{cf}(\tau) = \omega$ follows from (a). If $\rho < \tau$, then $2^\rho < 2^\tau$, since otherwise $\mathrm{Ps}(\tau, 2^\rho)$ (and consequently also $\mathrm{Ps}(\tau, 2^\tau)$) would hold in view of $\rho \leq \rho^\omega \leq \rho^+ \leq \tau \leq 2^\tau \leq 2^\rho$ and Lemma 1.5. This proves $\log 2^\tau \geq \tau$, the other inequality is trivial. Now assume $\mathrm{cf}(\tau) = \omega$ and $\log 2^\tau = \tau$. Hence in case $2^{<\tau} = 2^\tau$, Lemma 1.6(a) yields that $\mathrm{Ps}(\tau, 2^{<\tau})$ fails, hence $\mathrm{Ps}(\tau, 2^\tau)$ fails as well. In case $2^{<\tau} < 2^\tau$ note that $\Pi(\tau) \leq 2^{<\tau}$ by Lemma 1.6(a). Hence $\mathrm{Ps}(\tau, 2^\tau)$ fails again. □

1.2. Posets and their invariants

Here we give some information on posets which will be used in the sequel.

For a poset $(P, \leq)$ a *chain* is a totally ordered subset $\mathscr{C} \subseteq P$, $\mathscr{C}$ is *bounded* if $\mathscr{C}$ has a top element, otherwise $\mathscr{C}$ is *unbounded*. By *cofinality* of $\mathscr{C}$ we understand the least cardinality of a subset $\mathscr{C}'$ of $\mathscr{C}$ such that for each $c \in \mathscr{C}$ there is $c' \in \mathscr{C}'$ with $c \leq c'$.

In order to have a more precise language in discussing chain lengths into a poset $(P, \leq)$ define the following cardinal invariants

$$\mathrm{Ded}(P) = \min\{\lambda\colon \text{there is no chain of size } \lambda \text{ on } P\}$$

and

$$\mathrm{Ded_b}(P) = \min\{\lambda\colon \text{there is no bounded chain of size } \lambda \text{ on } P\}.$$

If P has a top element then obviously $\mathrm{Ded}(P) = \mathrm{Ded_b}(P)$. We write briefly $\mathrm{Ded}(\sigma)$ for $\mathrm{Ded}(\mathbf{P}(\sigma))$. This cardinal function was introduced by Shelah in a different context [37], obviously $\mathrm{Ded}(\sigma)$ is the least cardinal λ such that $C(\sigma, \lambda)$ fails.

Two partially ordered sets X and Y are *quasi-isomorphic* ($X \overset{\text{q.i.}}{\cong} Y$ in notation), if each one of them is isomorphic to a subset of the other [3, 16, 17]. Quasi-isomorphic posets obviously share a lot of common properties, such as monotone cardinal invariants, maximum size of well-ordered subsets, anti-well-ordered subsets, chains and anti-chains, etc. In particular for quasi-isomorphic posets P and Q we have $\mathrm{Ded}(P) = \mathrm{Ded}(Q)$ and $\mathrm{Ded_b}(P) = \mathrm{Ded_b}(Q)$. Note that $\mathbf{P}_\sigma(\kappa) \overset{\text{q.i.}}{\cong} \mathbf{P}_\sigma(\kappa')$ for some cardinals κ and κ' iff $\kappa = \kappa'$ [3, Corollary 2.4.].

1.11. Proposition. *Let* κ, κ', σ, σ', λ, λ' *be cardinals.*

(1) *If* $\mathscr{C}$ *is a chain of sets such that* $|C| \leq \kappa$ *for each* $C \in \mathscr{C}$, *then* $|\mathscr{C}| \leq 2^\kappa$. *In particular,* $C(\sigma, \lambda)$ *implies* $\lambda \leq 2^\sigma$ [8, Proposition 1.10].

(2) *Let* $\sigma \leq \sigma'$ *and* $\lambda' \leq \lambda$, *then* $C(\sigma, \lambda)$ *implies* $C(\sigma', \lambda')$ [2, Theorem 2.2(b)].

(3) *In ZFC* $C(\sigma,\sigma^+)$, $C(2^\sigma,2^{(\sigma^+)})$ *and* $C(\omega,2^\omega)$ *hold* [2, 8].

(4) *If* $\sigma < \kappa$, *then* $\mathrm{Ded}(\mathbf{P}_\sigma(\kappa)) = \mathrm{Ded}(\mathbf{P}_\sigma(\sigma^+))$ [3, Lemma 4.2].

(5) $\mathrm{Ded}(\sigma) \le \mathrm{Ded}(\mathbf{P}_\sigma(\sigma^+)) \le \mathrm{Ded}(\sigma)^+$; *more precisely*, $\mathrm{Ded}(\mathbf{P}_\sigma(\sigma^+)) = \mathrm{Ded}(\sigma)^+$ *iff* $\mathrm{cf}(\mathrm{Ded}(\sigma)) = \sigma^+$ [3, Theorem 6.10].

By (1) and (3) $\sigma^+ < \mathrm{Ded}(\sigma) \le (2^\sigma)^+$, so that under the assumption $2^\sigma = \sigma^+$ (in particular, under GCH) $C(\sigma,\lambda)$ is equivalent to $\lambda \le 2^\sigma = \sigma^+$ (i.e. $\mathrm{Ded}(\sigma) = (2^\sigma)^+ = \sigma^{++}$). In general $C(\sigma,2^\sigma)$ may fail (i.e. $\mathrm{Ded}(\sigma) = 2^\sigma$ may occur). For an example with $\sigma = \omega_1$, see [31]. It follows from (5) that $\mathrm{Ded}(\mathbf{P}_\sigma(\sigma^+)) = \mathrm{Ded}(\sigma)$ under GCH, but in general this equality cannot be determined in ZFC even for $\sigma = \omega_1$ [3, Corollary 5.7].

1.3. Topological groups

We denote by $\mathbb{N}$ the naturals, by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the circle group. We use $r(G)$ to denote the free-rank of an Abelian group G and as usual denote by $r_p(G)$ the p-rank of G. For a discrete group G we denote by $C\mathscr{P}(G)$ (resp. $\mathscr{Z}\mathscr{P}(G)$) the poset of connected (resp. zero-dimensional) topologies in $\mathscr{P}(G)$ and for a cardinal σ we set $\mathscr{Z}\mathscr{P}_\sigma(G) = \mathscr{P}_\sigma(G) \cap \mathscr{Z}\mathscr{P}(G)$ and $C\mathscr{P}_\sigma(G) = \mathscr{P}_\sigma(G) \cap C\mathscr{P}(G)$. Further, we denote by $b_G : G \to G^\#$ the Bohr compactification of G; the group G is *maximally almost periodic* if b_G is a monomorphism, i.e. $\mathscr{B}(G) \neq \emptyset$ [43]. Clearly, $\mathfrak{P}$ consists of maximally almost periodic groups. Following [16], for a maximally almost periodic group G we set $\gamma(G) = \min\{\sigma : \mathscr{B}_\sigma(G) \neq \emptyset\}$ and $\Gamma(G) = \sup\{\sigma\colon \mathscr{B}_\sigma(G) \neq \emptyset\}$. Now we introduce the counterpart of these invariants in the pseudocompact case.

1.12. Definition. For a group $G \in \mathfrak{P}$ let

$$\pi(G) = \min\{\sigma\colon\ G \in \mathfrak{P}(\sigma)\} \quad \text{and} \quad \Pi(G) = \sup\{\sigma\colon\ G \in \mathfrak{P}(\sigma)\}.$$

In general

$$\log|G| \le \gamma(G) \le \pi(G) \le \Pi(G) \le \Pi(|G|) \le \Gamma(G) \le 2^{|G|},$$

and the equalities $\log|G| = \gamma(G) = \pi(G)$ and $\Pi(G) = \Pi(|G|)$ obviously hold for a CR-group G (this shows that in the non-Abelian case CR-groups satisfy the stringent condition $\log|G| = \gamma(G)$ which may fail for a maximally almost periodic group G, actually $\gamma(G)$ may take all possible values between $\log|G|$ and $|G|$: [3,7.13; 16, p. 145]). In these terms Question 0.1(b) asks whether the equality $\log|G| = \pi(G)$ remains true for all groups $G \in \mathfrak{P}$. We show that the answer is "yes" for the groups considered in this paper (relatively free and residually finite, torsion Abelian, torsion-free Abelian, divisible Abelian, etc.). On the other hand, we see in Section 5 that $\Pi(G) < \Pi(|G|)$ (and actually, $2^{\Pi(G)} \le \Pi(|G|)$) may occur (see Theorem 5.7). An easy modification of the proof of [3, Lemma 7.3(1)] shows that if $\pi(G) < \Pi(G)$, then $G \in \mathfrak{P}(\sigma)$ for every $\sigma \in [\pi(G), \Pi(G))$.

1.13. Fact. *For every group G there exist a set X and an order-isomorphism φ of $\mathscr{B}(G)$ into $\mathbf{P}(X)$, which sends $\mathscr{B}_\sigma(G)$ into $\mathbf{P}_\sigma(X)$. In particular, if G admits a chain with top element of λ precompact group topologies of weight σ, then $C(\sigma,\lambda)$ holds.*

Proof. For the existence of φ see [36, Theorem 2.3]. The last assertion is obvious. □

1.14. Remark. (a) In [36, Theorem 2.3] the set X is the Tanaka–Krein dual of the Bohr compactification $G^\#$ of the discrete group G. Note that, in view of the monotonicity of φ, Fact 1.13 yields $w(G,T) \leq w(G,T')$ whenever $T \leq T'$ in $\mathscr{B}(G)$.

(b) Since the depth of $\mathbf{P}_\sigma(X)$ is σ, Fact 1.13 yields immediately that $\mathscr{B}_\sigma(G)$ cannot contain copies of $\mathbf{P}(\kappa)$ for $\kappa > \sigma$.

(c) If σ is an infinite cardinal such that $\mathscr{B}_\sigma(G) \neq \emptyset$ for some group G, then $\log|G| \leq \sigma \leq 2^{|G|}$. Actually, this inequality is true for any Hausdorff topological space G of weight σ.

(d) Boundedness of the chain is essential in Fact 1.13. In fact, it may happen that $\mathscr{B}_\sigma(G)$ or even $\mathscr{P}_\sigma(G)$ contain a chain of size λ such that $C(\sigma,\lambda)$ does not hold (see Section 6).

(e) Precompactness is essential in Fact 1.13. In fact, for $G = \bigoplus_\omega \mathbb{Z}(2)$ the poset of linear metrizable group topologies on G does not embed in any $\mathbf{P}_\omega(X)$ [3].

In the sequel we show that in many cases a copy of $\mathbf{P}(\sigma)$ is contained even in $\mathscr{P}_\sigma(G)$ or some subsets of that poset. This will determine the possible lengths λ of bounded chains in $\mathscr{P}_\sigma(G)$, namely, all $\lambda < \mathrm{Ded_b}(\mathscr{P}_\sigma(G)) = \mathrm{Ded}(\sigma)$.

A topological space X is *locally connected* if for every point $x \in X$ each neighbourhood of x contains an open neighbourhood of x with connected closure. For a group G we denote by $CLC\mathscr{P}(G)$ the poset of connected and locally connected topologies in $\mathscr{P}(G)$ and for a cardinal σ we set $CLCP_\sigma(G) = \mathscr{P}_\sigma(G) \cap CLCP(G)$.

We introduce also the following notion of disconnectedness for topological groups. A topological group G is *totally l-disconnected* if the only locally connected subgroups of G are the discrete ones. In our outline we shall meet many connected and totally l-disconnected pseudocompact groups, so this will justify our attention to this property. It is easy to see, applying Pontryagin duality, that among metrizable compact Abelian groups the totally l-disconnected ones are precisely those which have no topological direct summand isomorphic to the circle group $\mathbb{T}$.

The following lemma is the first step towards constructing large sets of pseudocompact group topologies on a given group G.

1.15. Lemma. *Let G be a dense pseudocompact subgroup of a compact group K with $w(K) = \sigma$. Assume that there exists a family $\mathcal{N} = \{N_i : i \in I\}$ of closed normal subgroups of K which satisfy:*

(i) *for each $i \in I$ $G \cap N_i = \{e\}$;*

(ii) *for each $i \in I$ $w(K/N_i) = \sigma$.*

Let $\mathscr{E}$ be a topological property of pseudocompact groups preserved by taking dense subgroups. If K has $\mathscr{E}$ then $\mathscr{P}_\sigma(G)$ contains a subset $\mathscr{T}$ anti-isomorphic to $\mathscr{N}$ consisting of topologies with the property $\mathscr{E}$.

Proof. For $i \in I$ denote by φ_i the canonical homomorphism $K \to K/N_i$. By (i) the restriction of φ_i on G is a monomorphism, thus φ_i induces on G a coarser group topology τ_i, namely the topology induced on $\varphi_i(G)$ from the quotient K/N_i. By the property (ii) it follows that for each $i \in I$ τ_i is a group topology of weight σ. Moreover, $\tau_i \le \tau_{i'}$ whenever $N_{i'} \subseteq N_i$. Therefore $\mathscr{T} = \{\tau_i \colon i \in I\}$ is anti-isomorphic to $\mathscr{N}$.

Now assume that K has $\mathscr{E}$. Then each K/N_i has $\mathscr{E}$. For each $i \in I$ (G, τ_i) is a dense pseudocompact subgroup of K/N_i, so has $\mathscr{E}$ as well. □

1.16. Lemma. *Each of the following properties is preserved under taking dense pseudocompact subgroups*: (a) *connectedness*; (a*) *disconnectedness*; (b) *local connectedness*; (b*) *non-local-connectedness*; (c) *zero-dimensionality*; (d) *total l-disconnectedness. Moreover*, (a), (b) *and* (c) *are preserved under products.*

Proof. (a*), (b*) and the last assertion are well known. For a pseudocompact group G the completion $\hat{G}$ coincides with the Čech–Stone compactification βG of the group G [26]. This proves items (a) and (b). Finally, (c) and (d) are obvious. □

2. Relatively free groups

By a *variety of groups* we mean, as usual, a class $\mathfrak{V}$ of groups closed under Cartesian products, subgroups and quotients (see [33] for further information). We call *quasi-variety of groups* a class $\mathfrak{V}$ of groups closed under Cartesian products and subgroups. For a class of groups $\mathfrak{V}$ we denote by $\mathrm{var}(\mathfrak{V})$ (resp. $\mathrm{qvar}(\mathfrak{V})$) the variety (resp. quasi-variety) generated by $\mathfrak{V}$. In case $\mathfrak{V} = \{H\}$ we write simply $\mathrm{var}(H)$ or $\mathrm{qvar}(H)$. We use $\mathfrak{G}$ and $\mathfrak{A}$ for denoting the varieties of all groups and all Abelian groups, respectively. Every quasi variety $\mathfrak{V}$ determines a reflector $\rho_{\mathfrak{V}} : \mathfrak{G} \to \mathfrak{V}$, i.e. an endofunctor $\rho_{\mathfrak{V}} : \mathfrak{G} \to \mathfrak{G}$ such that for every $G \in \mathfrak{G}$ there exists a surjective homomorphism $\rho^G_{\mathfrak{V}} : G \to \rho_{\mathfrak{V}}(G) \in \mathfrak{V}$ with the obvious universal property (just take the quotient homomorphism with kernel the intersection of all kernels of homomorphisms $G \to G_1 \in \mathfrak{V}$).

Let $\mathfrak{V}$ be a variety. The *$\mathfrak{V}$-free* group of τ generators, denoted by $F_\tau(\mathfrak{V})$, is a group $G \in \mathfrak{V}$ having a set X of τ generators such that every *map* $f : X \to H \in \mathfrak{V}$ can by uniquely extended to a *homomorphism* $\hat{f} : G \to H$. For $\mathfrak{V} = \mathfrak{G}$ we write simply F_τ instead of $F_\tau(\mathfrak{G})$. In the general case, one can take as $F_\tau(\mathfrak{V})$ the group $\rho_{\mathfrak{V}}(F_\tau)$. A group G is *relatively free* if G is $\mathrm{var}(G)$-free.

In this section we show, roughly speaking, that lengths of chains with top element of pseudocompact topologies of weight σ on a free Abelian groups are the same as the lengths of chains in $\mathbf{P}(\sigma)$. More precisely, for an arbitrary variety $\mathfrak{V}$ of groups we completely solve the following problem: When does the $\mathfrak{V}$-free group $F_\tau(\mathfrak{V})$ admit a chain, with top element and length λ, of pseudocompact group topologies of weight σ? We consider a property of a variety $\mathfrak{V}$ (see Definition 2.1 below) which turns out to be necessary for the positive solution of this problem for *some* cardinals τ and λ [23, Theorem 5.5]. On the other hand, if this necessary condition holds for $\mathfrak{V}$, then the answer to the question depends only on the purely set-theoretic properties $\mathrm{Ps}(\tau,\sigma)$ and $C(\sigma,\lambda)$. In the case $\mathfrak{V} \in \{\mathfrak{G},\mathfrak{A}\}$ this condition is satisfied, and $F_\tau(\mathfrak{V})$ admits long chains with top element of pseudocompact connected and locally connected group topologies of weight σ whenever $\mathrm{Ps}(\tau,\sigma)$ holds.

For a variety $\mathfrak{V}$ let $F_1,F_2,\dots,F_n,\dots$ be a full list of pairwise nonisomorphic finite groups of $\mathfrak{V}$. The infinite group $K_{\mathfrak{V}} = \prod_{n=1}^{\infty} F_n \in \mathfrak{V}$ will always be equipped with the product topology of the discrete topologies on each F_n. Note that the topology of $K_{\mathfrak{V}}$ is compact, metrizable and linear, in particular, zero-dimensional. Set $\mathfrak{V}^f := \mathrm{var}(K_{\mathfrak{V}})$.

2.1. Definition. A variety $\mathfrak{V}$ is *precompact* if $\mathfrak{V} = \mathfrak{V}^f$, i.e. $\mathfrak{V}$ is generated by its finite groups.

A group G is *residually finite* if the intersection of all its normal subgroups of finite index is trivial, or equivalently, if G is isomorphic to a subgroup of a Cartesian product of finite groups [33, 17.71–17.73]. Hence every residually finite group is maximally almost periodic. It turns out that for a relatively free group G these two properties are equivalent. In fact, if a variety $\mathfrak{V}$ is precompact, then every $\mathfrak{V}$-free group is isomorphic to a subgroup of a power of $K_{\mathfrak{V}}$, hence admits a precompact zero-dimensional group topology (this need not be true for *all* groups of $\mathfrak{V}$, see Remark 2.8(c) and Remark 3.5(b), (c)). Conversely, these are precisely the varieties $\mathfrak{V}$ for which every finitely generated $\mathfrak{V}$-free group admits precompact group topology. Hence, a variety $\mathfrak{V}$ is precompact iff $\mathfrak{V}$ is generated by its compact groups. For a proof of these facts see [16] or [23].

The varieties $\mathfrak{G}$ and $\mathfrak{A}$ as well as many of the known varieties are precompact (nilpotent groups, polynilpotent groups, soluble groups, in particular, metabelian groups). Nonprecompact varieties are not easy to come by, in fact, for prime $p > 665$ the Burnside variety $\mathfrak{B}_p$ (consisting of all groups satisfying the identity $x^p = 1$) is not precompact [16]. The proof is based on the negative solution of Burnside's problem [33], the positive solution of the restricted Burnside's problem [47] and the fact that bounded torsion subgroups of finite-dimensional unitary groups are finite. In particular:

$\mathfrak{B}_n$ is precompact iff Burnside's problem for n has a positive solution.

Consequently, the varieties $\mathfrak{B}_2$, $\mathfrak{B}_3$, $\mathfrak{B}_4$ and $\mathfrak{B}_6$ are precompact.

It was proved in [23] that for a precompact variety $\mathfrak{V}$ and cardinals τ and $\sigma \geq \omega_1$ given the group $F_\tau(\mathfrak{V})$ admits a pseudocompact group topology of weight σ iff $\mathrm{Ps}(\tau,\sigma)$ holds. The following lemma turns out to be fundamental for our approach to chains of pseudocompact topologies on groups. Only the part in parenthesis will be used in this paper, we give the other version only for the sake of completeness.

2.2. Lemma. *Let $\mathfrak{V}$ be a precompact variety of groups, let H be an infinite compact metrizable group generating $\mathfrak{V}$ and let $L \in \mathfrak{V}$ be a compact group of weight $\leq \sigma$. Suppose that τ and $\sigma \geq \omega_1$ are cardinals, such that* $\log\sigma \leq \tau \leq 2^\sigma$ *(resp.* $\mathrm{Ps}(\tau,\sigma)$*) hold. Then there exist a dense (pseudocompact) $\mathfrak{V}$-free subgroup $F = F_\tau(\mathfrak{V})$ of $H^\sigma \times L$ and a family $\mathcal{N} = \{N_i : i \in I\}$ of closed normal subgroups of $H^\sigma \times L$ which satisfy*

(i) $F \cap N_i = \{e\}$;

(ii) *each N_i is a topological direct summand of $H^\sigma \times L$;*

(iii) *for each $i \in I$ $(H^\sigma \times L)/N_i \cong H^\sigma \times L$ and $N_i \cong H^\sigma$, in particular $w(H^\sigma \times L/N_i) = \sigma$;*

(iv) *the family $\mathcal{N} = \{N_i\colon i \in I\}$ is order-isomorphic to* $\mathbf{P}(\sigma)$.

Proof. Assume $\mathrm{Ps}(\tau,\sigma)$ holds. According to [23, Lemma 4.3] (see also [21]) and [23, Remark 9.15] there exist an ω-dense $\mathfrak{V}$-free subgroup F of $H^\sigma \times L$ with $|F| = \tau$ and a set $C \subseteq \sigma$ such that $|C| = |\sigma \setminus C| = \sigma$ and $F \cap H^C = \{e\}$, where $H^C = \{h \in H^\sigma\colon h(\alpha) = e$ for all $\alpha \in \sigma \setminus C\}$ and e is the neutral element of H. This will do for the proof of the part in parenthesis. For the other part one can prove analogously, by substituting in the proof of [23, Lemma 4.3], wherever necessary, the condition $\mathrm{Ps}(\tau,\sigma)$ by $\log\sigma \leq \tau \leq 2^\sigma$ and projections on countable subproducts H^A by projections on finite subproducts the existence of a *dense* $\mathfrak{V}$-independent subset X with the same properties.

Now fix a subset B of C with $|C \setminus B| = |B| = \sigma$ and set $N_D = H^D$ for $B \subseteq D \subseteq C$. Then obviously the family $\mathcal{N} = \{N_D\colon B \subseteq D \subseteq C\}$ of closed normal subgroups of H^σ satisfies $F \cap N_D = \{e\}$, i.e. (i) holds. Further, the family $\mathcal{N}$ is order-isomorphic to $\mathbf{P}(C \setminus B) = \mathbf{P}(\sigma)$ with respect to the inclusion, i.e. (iv) holds with $I = \{D\colon B \subseteq D \subseteq C\}$. By the choice of C and B $(H^\sigma \times L)/N_D \cong H^{\sigma\setminus D} \times L \cong H^\sigma \times L$ and $N_D \cong H^\sigma$, so (iii) holds. Finally, (ii) follows directly form the definition of N_D. □

The equivalence of items (iv) and (v) in the following theorem was proved in [20] and [23, Theorem 5.6].

2.3. Theorem. *Let $\mathfrak{V}$ be a precompact variety and let τ and $\sigma \geq \omega_1$ be cardinals. Then the following conditions are equivalent for the $\mathfrak{V}$-free group $G = F_\tau(\mathfrak{V})$.*

(i) *the poset $\mathcal{X}\mathcal{P}_\sigma(G)$ contains a copy of the Boolean algebra* $\mathbf{P}(\sigma)$;

(ii) *$\mathcal{P}_\sigma(G)$ contains a copy of the Boolean algebra* $\mathbf{P}(\sigma)$;

(iii) $\mathrm{Ded_b}(\mathcal{P}_\sigma(G)) \geq \mathrm{Ded}(\sigma)$ *(i.e. $\mathcal{P}_\sigma(G)$ contains a bounded chain of length λ for each λ satisfying $C(\sigma,\lambda)$)*;

(iii*) $\mathrm{Ded_b}(\mathcal{P}_\sigma(G)) = \mathrm{Ded}(\sigma)$;

(iv) $\mathcal{P}_\sigma(G) \neq \emptyset$;

(v) $\mathrm{Ps}(\tau,\sigma)$ *holds.*

In case $\mathfrak{V} = \mathfrak{A}$ or $\mathfrak{V} = \mathfrak{G}$ these conditions are equivalent to the following:

(vi) *$\mathscr{P}_\sigma(G)$ contains a copy of the Boolean algebra $\mathbf{P}(\sigma)$ consisting of topologies with one of the following three properties: connected and locally connected, disconnected and locally connected, connected and nonlocally-connected.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iii*) $\Rightarrow$ (iv) and (vi) $\Rightarrow$ (ii) are obvious. The implication (iv) $\Rightarrow$ (v) follows from Fact 1.1. The implication (iii) $\Rightarrow$ (iii*) follows from Fact 1.13. To finish the proof we have to prove the implication (v) $\Rightarrow$ (i) and the implication (v) $\Rightarrow$ (vi) in the case $\mathfrak{V} = \mathfrak{A}$ or $\mathfrak{V} = \mathfrak{G}$.

By the precompactness condition the variety $\mathfrak{V}$ is generated by a compact group $H \in \mathfrak{V}$, which may be assumed zero-dimensional if necessary.

Assume $\mathrm{Ps}(\tau, \sigma)$ holds, then by Lemma 2.2 (applied with $L = \{0\}$) there exists a family $\mathscr{N} = \{N_i : i \in I\}$ of closed normal subgroups of H^σ which satisfy conditions (i) and (ii) from Lemma 1.15. Since H was assumed zero-dimensional and zero-dimensionality is a property preserved by taking arbitrary subgroups and products, the power H^σ is zero-dimensional. We can apply now for this property Lemma 1.15 to get a family $\{\tau_i \colon i \in I\} \cong \mathbf{P}(\sigma)$ of zero-dimensional pseudocompact group topologies of weight σ on G. This proves the implication (v) $\Rightarrow$ (i).

Now consider the case $\mathfrak{V} = \mathfrak{A}$ or $\mathfrak{G}$. It is known that both varieties are generated by connected and locally connected groups H. For $\mathfrak{A}$ take $H = \mathbb{T}$, for $\mathfrak{G}$ take for example the orthogonal group $\mathrm{SO}(3, \mathbb{R})$ [1]. Note that each of the following four properties of pseudocompact topological groups: connectedness, disconnectedness, local connectedness and nonlocal connectedness are preserved by taking dense pseudocompact subgroups according to Lemma 1.16. Moreover, both connectedness and local connectedness are preserved by products. Thus in both cases considered the groups H^σ may be chosen to be both connected and locally connected. Apply now for this property Lemma 1.15 to get a family $\{\tau_i \colon i \in I\} \cong \mathbf{P}(\sigma)$ of connected and locally connected pseudocompact group topologies of weight σ on G. For the case $\mathscr{E}$ is the property "disconnected and locally-connected" take a finite nontrivial group L in $\mathfrak{V}$ and instead of H^σ take the group $H^\sigma \times L$. Now apply Lemma 2.2 to this data to get a family $\mathscr{N} = \{N_i \colon i \in I\}$ of subgroups of $H^\sigma \times L$ satisfying conditions (i)–(iv) of Lemma 2.2. Note that now $H^\sigma \times L$ is disconnected and locally connected, so that by virtue of the isomorphism (iii) in Lemma 2.2 also the quotient group $H^\sigma \times L/N_i \cong H^\sigma \times L$ has the same property. An application of (the proof of) Lemma 1.15 to the family $\mathscr{N}$ will produce a family $\{\tau_i \colon i \in I\} \cong \mathbf{P}(\sigma)$ of disconnected and locally connected pseudocompact group topologies of weight σ on G.

Let now $\mathscr{E}$ be the property "connected and nonlocally-connected". To be in position to apply again Lemma 1.15 we need to choose H to have the same property. Let $D = \mathbb{Q}^*$ be Pontryagin dual of the rationals. Then D is a compact connected (hence divisible) metrizable Abelian group which is not locally connected [19, Chapter 3]. Now take $H = \mathrm{SO}(3, \mathbb{R}) \times D$ in the case $\mathfrak{V} = \mathfrak{G}$, and take $H = D$ in case $\mathfrak{V} = \mathfrak{A}$. $\square$

2.4. Corollary. *Let $\mathfrak{B}$ be a precompact variety. Then for cardinals τ, λ and $\sigma \geq \omega_1$ given, the following conditions are equivalent*:

(i) $\mathrm{Ps}(\tau, \sigma)$ *and* $C(\sigma, \lambda)$ *hold*;

(ii) $F_\tau(\mathfrak{B})$ *admits a chain of length λ, with both bottom and top elements, of pseudocompact zero-dimensional group topologies of weight σ*;

(iii) $F_\tau(\mathfrak{B})$ *admits a chain of length λ with top element of precompact group topologies of weight σ and* $\mathrm{Ps}(\tau, \sigma)$ *holds.*

Moreover, in case $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{G}$ these conditions are equivalent to the following:

(iv) $F_\tau(\mathfrak{B})$ *admits a chain of length λ, with both bottom and top elements, of pseudocompact connected and locally connected group topologies of weight σ.*

Proof. The implications (i) $\Rightarrow$ (ii) and, in case $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{G}$, (i) $\Rightarrow$ (iv) are trivial applications of Theorem 2.3. The implication (ii) $\Rightarrow$ (iii) follows from Fact 1.1. The implications (iii) $\Rightarrow$ (i) and, in case $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{G}$, (iv) $\Rightarrow$ (i) follow from Fact 1.13. □

2.5. Remark. (a) In case $F_\tau(\mathfrak{B})$ admits an arbitrary chain (i.e. without the assumption of boundedness) of precompact group topologies of weight σ and $\mathrm{Ps}(\tau, \sigma)$ holds, then all we can get from this is that $\lambda \leq 2^\sigma$ (by Fact 1.13 and Proposition 1.11). More precisely, one can show that there exists a chain of length λ in $\mathbf{P}_\sigma(\sigma^+)$ which is a less stringent condition than $C(\sigma, \lambda)$ (see Proposition 1.11(5) and Theorem 6.3).

(b) The restraint $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{G}$ for the validity of the implication (i) $\Rightarrow$ (iv) is necessary – it is proved in [23] that if $F_\tau(\mathfrak{B})$ admits a pseudocompact connected group topology then either $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{B} = \mathfrak{G}$.

Now we can answer Question 0.1 in the case of relatively free groups.

2.6. Corollary. *Every relatively free group in $\mathfrak{P}$ of cardinality $> \mathfrak{c}$ is a* CR-*group.*

It follows from Lemma 1.2(b) that if $\mathrm{Ps}(\tau, \sigma)$ holds for some σ then also $\mathrm{Ps}(\tau, \log \tau)$ holds. Thus according to the above corollary, if $\mathscr{P}(G) \neq \emptyset$ for a relatively free group G with $|G| > \mathfrak{c}$, then also $\mathscr{P}_{\log|G|}(G) \neq \emptyset$ (i.e. $\pi(G) = \log|G|$) as in the case of precompact topologies.

Now we give an example of a free Abelian group to show that as far as pseudocompactness is concerned the maximal weight $2^{|G|}$ need not be attained, i.e. $\mathscr{P}_{2^{|G|}}(G) = \emptyset$ (even $\Pi(G)^+ = 2^{|G|}$) may occur while $\mathscr{P}(G) \neq \emptyset$ and $\mathscr{B}_{2^{|G|}}(G) \neq \emptyset$.

2.7. Example. Let $G = \mathbb{Z}^{(\omega_\omega)}$. There is a model, $\mathfrak{M}$ of ZFC such that:

(i) for each $n \in \mathbb{N}^+$ the group G has a pseudocompact group topology of weight 2^{ω_n} in $\mathfrak{M}$, hence $\Pi(G) \geq 2^{<|G|}$ (actually, $\Pi(G)^+ = 2^{|G|}$);

(ii) G has pseudocompact group topologies of weight neither $2^{|G|}$ nor $\sup 2^{\omega_n}$ in $\mathfrak{M}$, thus $\Pi(G) = 2^{<|G|}$.

According to Easton's theorem [25] there exists a model $\mathfrak{M}$ of ZFC such that the following rules of cardinal arithmetic hold in $\mathfrak{M}$:

$$2^{\omega_n} = \omega_{\omega+n}, \quad 2^{\omega_{\omega+n}} = \omega_{\omega+\omega+n}, \quad 2^{\omega_{\omega+\omega+n}} = \omega_{\omega+\omega+\omega+1} \qquad \text{for every } n \in \mathbb{N}^+,$$

$$2^{\omega_\omega} = \omega_{\omega+\omega+1}, \qquad 2^{\omega_{\omega+\omega}} = \omega_{\omega+\omega+\omega+1}, \qquad 2^\kappa = \kappa^+ \quad \text{for every } \kappa \geq \omega_{\omega+\omega+\omega}.$$

Note that the model $\mathfrak{M}$ satisfies SCH, so that $\mathrm{cf}(|G|) = \omega$ and $2^{<|G|} = \omega_{\omega+\omega}$ (a proper limit) yield $\Pi(|G|) = 2^{<|G|}$ by Corollary 1.10. Hence for each $n \in \mathbb{N}^+$ the cardinal $|G| = \omega_\omega$ is $2^{\omega_n} = \omega_{\omega+n}$-admissible in $\mathfrak{M}$ (this proves (i)), while $\mathrm{Ps}(|G|, 2^{<|G|})$ fails so that $|G|$ is neither $\omega_{\omega+\omega}$-admissible, nor $2^{|G|}$-admissible (this proves (ii)). □

2.8. Remark. (a) The above example shows that in the model $\mathfrak{M}$ the subset $\mathscr{P}(G)$ of $\mathscr{B}(G)$ is not closed even under countable directed suprema. In fact, let for each $n \in \mathbb{N}$ $T_n \in \mathscr{P}_{\omega_{\omega+n}}(G)$. Then $\omega_{\omega+\omega} = \sup\{2^{\omega_n} : n \in \mathbb{N}\}$ yields $T = \sup\{T_n\} \in \mathscr{B}_{\omega_{\omega+\omega}}(G)$, while (ii) says that $\mathscr{P}_{\omega_{\omega+\omega}}(G) = \emptyset$. Hence T is not pseudocompact. It was kindly pointed out by the referee that an increasing sequence of pseudocompact group topologies on $\mathbb{Z}^{(\mathfrak{c})}$ such that the supremum of this sequence is not pseudocompact has been produced by Tkačenko [39, Example 4.2] in ZFC.

(b) In the above example the group G can be taken any Abelian CR-group, for example G which satisfies $|G| = r(G)$ (see Corollary 3.4) or $G = F^{(\omega_\omega)}$, where F is a finite Abelian group (see Example 4.6(b); G can be also a tame bounded p-group as defined in the proof of Theorem 4.1).

(c) The following example shows the importance of relative freeness in 2.3 and 2.4. Let F be a finite simple non-Abelian group and $\mathfrak{V} = \mathrm{var}(F)$. Then $\mathfrak{V}$ is precompact, so that by Theorem 2.3 the $\mathfrak{V}$-free groups of admissible cardinality admit zero-dimensional pseudocompact group topologies, but the group $G = \bigoplus_\tau F$ admits pseudocompact group topologies for no infinite τ. In fact, $\mathscr{B}(G) = \{T_f\}$, where T_f is the product topology of G (see [3, Example 7.13]. Since G is not G_δ-dense in its T_f-completion F^τ, T_f is not pseudocompact.

3. Precompact varieties with injective compact generator

3.1. Definition. Let $\mathfrak{V}$ be a variety of groups and $H \in \mathfrak{V}$. We say that H is $\mathfrak{V}$-*injective* if for every $G \in \mathfrak{V}$ every homomorphism $f : G_1 \to H$, where G_1 is a subgroup of G, has an extension $\tilde{f} : G \to H$.

3.2. Proposition. *Let $\mathfrak{V}$ be a variety of Abelian groups. Then*:

(a) *There exists an infinite compact metrizable $\mathfrak{V}$-injective generator H of $\mathfrak{V}$.*

(b) *Let H be as in* (a), *let $\sigma \geq \omega_1$ be a cardinal and let $G \in \mathfrak{V}$ contain a $\mathfrak{V}$-free subgroup F with* $\mathrm{Ps}(|F|, \sigma)$. *Then the following are equivalent*:

(b_1) $|G| \leq 2^\sigma$ *and for each prime* $p \in \mathbb{P}$ $r_p(G) > 0$ *yields* $r_p(H) > 0$ (*i.e.* $G \in \mathrm{qvar}(H)$),

(b_2) *there exist a dense pseudocompact subgroup* G_1 *of* H^σ *isomorphic to* G *and a family* $\mathcal{N} = \{N_i\colon i \in I\}$ *of closed topological direct summands of* H^σ *which satisfy* $G_1 \cap N_i = \{e\}$, $w(H^\sigma/N_i) = \sigma$ *for each* $i \in I$ *and* $\mathcal{N}$ *is order-isomorphic to* $\mathbf{P}(\sigma)$. *Both* (b_1) *and* (b_2) *yield* $\mathbf{P}(\sigma) \hookrightarrow \mathscr{P}_\sigma(G)$.

Proof. (a) Let us note first of all that the variety $\mathfrak{B}$ is precompact. Hence there exists an infinite compact metrizable group H generating $\mathfrak{B}$. Set now $\mathfrak{A}_n = \mathfrak{B}_n \cap \mathfrak{A}$ and note that either $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{B} = \mathfrak{A}_n$ for some $n \in \mathbb{N}$. In the case $\mathfrak{B} = \mathfrak{A}$ we can take $H = \mathbb{T}$. Being divisible this group is also injective in $\mathfrak{A}$. In the case $\mathfrak{B} = \mathfrak{A}_n$ for some $n \in \mathbb{N}$ take $H = \mathbb{Z}(n)^\omega$. To see that this is an injective object of $\mathfrak{A}_n$ it suffices to check that the group $\mathbb{Z}(n)$ is an injective object of $\mathfrak{A}_n$. Let $G \in \mathfrak{A}_n$ and let $f: G_1 \to \mathbb{Z}(n)$ be a homomorphism, where G_1 is a subgroup of G. It suffices to extend f to any subgroup $G_2 = G_1 + \langle x\rangle$ of G with $x \notin G_1$ and then apply transfinite induction. If the sum is direct we are through. Assume that $G_1 \cap \langle x\rangle = \langle kx\rangle$ for some $k \in \mathbb{N}$ with $kx \neq 0$. It is not restrictive to assume that $1 < k < n$ and k is a divisor of n. Let $n = km$. Since $nx = 0$, the order of the element $kx \in G_1$ is a divisor of m. Thus $mf(kx) = 0$ in $\mathbb{Z}(n)$. Since $\mathbb{Z}(n)[m] = k\mathbb{Z}(n)$ there exists $a \in \mathbb{Z}(n)$ such that $f(kx) = ka$. Now set $\tilde{f}(x) = a$ and extend $\tilde{f}$ to G_2 by linearity.

To prove (b) note that the implication (b_2) $\Rightarrow$ (b_1) is trivial. We proceed now with the proof of the implication (b_1) $\Rightarrow$ (b_2). According to Lemma 2.2 there exists a dense pseudocompact $\mathfrak{B}$-free subgroup F' of H^σ with $|F'| = |F|$ and a family $\mathcal{N} = \{N_i : i \in I\}$ of closed subgroups of H^σ which satisfy conditions (i)–(iv) from Lemma 2.2. Let T denote the top element of the family $\mathcal{N}$. In particular, we have

$$F' \cap T = \{0\}. \tag{1}$$

We intend to "extend" the inclusion of F' into H^σ to an embedding $\tilde{\jmath} : G \to H^\sigma$ such that for $G_1 = \tilde{\jmath}(G)$ the equality $G_1 \cap T = \{0\}$ holds. Then obviously also $G_1 \cap N_i = \{0\}$ will be satisfied for $i \in I$. Let T_o be the bottom element of the family $\mathcal{N}$. Then $T = T_o \times T_1$, where $T_o \cong T_1 \cong H^\sigma$ by (iii) of Lemma 2.2.

Set $L = F' + T_o$ and note that this sum is direct by (1). Let us prove that (by the modular law applied to the lattice of all subgroups of the group H^σ)

$$L \cap T_1 = \{0\}. \tag{2}$$

In fact, if $x = f + t \in T_1 \cap L$, $f \in F'$, $t \in T_o$, then $f = x - t \in F' \cap (T_o + T_1)$. By (1) this yields $f = x - t = 0$ and consequently $x = t \in T_o \cap T_1 = \{0\}$. Hence $x = 0$. This proves (2). Now fix an isomorphism $j : F \to F'$ between the $\mathfrak{B}$-free groups F and F'. In case $\mathfrak{B} = \mathfrak{A}$ consider also a free subgroup F_G of G such that $F \cap F_G = \{0\}$ and such that $G/(F + F_G)$ is torsion. In case $\mathfrak{B} \neq \mathfrak{A}$ set $F_G = 0$. By $|F_G| \leq |G| \leq 2^\sigma$ there exists a monomorphism $d : F_G \to T_o$. With $\tilde{F} = F + F_G$, in both cases the subgroup

$G' = \tilde{F} + s(G)$ is an essential subgroup of G (i.e. nontrivially meets every nontrivial subgroup of G). Moreover, the sum $\bar{\jmath} = j + d : \tilde{F} \to L$ is a monomorphism in view of (1) and the choice of the homomorphism d.

Assume $r_p(G) > 0$ for some $p \in \mathbb{P}$. The subgroup $\tilde{F}[p]$ splits off in $G[p]$, i.e. $G[p] = \tilde{F}[p] \oplus S_p$ (in case $\mathfrak{B} = \mathfrak{A}$, $\tilde{F}[p] = 0$ thus $S_p = G[p]$). By hypothesis also $r_p(H) > 0$. By $|G| \leq 2^\sigma$ it follows that $r_p(G) \leq 2^\sigma$. Thus there exists a monomorphism $i_p : S_p \to s_p(T_o) \cong s_p(H^\sigma) = s_p(H)^\sigma$. Note that the sum $j_p = \bar{\jmath} \restriction_{\tilde{F}[p]} + i_p : G[p] = \tilde{F}[p] \oplus S_p \to H^\sigma$ is a monomorphism for each $p \in \mathbb{P}$ (if $\mathfrak{B} = \mathfrak{A}$, then $\tilde{F}[p] = 0$; otherwise $\tilde{F} = F$, so that $\bar{\jmath}(\tilde{F}) = F'$ and (1) applies). Therefore, there exists a monomorphism $j' : s(G) \to L^\sigma$ which extends the restriction $\bar{\jmath} \restriction_{s(\tilde{F})} : s(\tilde{F}) \to H^\sigma$. Let $j'' : G' = \tilde{F} + s(G) \to H^\sigma$ be the sum of $\bar{\jmath}$ and j'. For each nonzero $x \in \tilde{F} + s(G)$ there exists $k \in \mathbb{N}$ such that $y = kx \neq 0$ and either $y \in \tilde{F}$ or $y \in s(G)$. In both cases $j''(y) \neq 0$, so that $j''(x) \neq 0$ as well. Since H^σ is $\mathfrak{B}$-injective, there exists an extension $\tilde{\jmath} : G \to H^\sigma$ of j''. Then also $\tilde{\jmath}$ is a monomorphism since G' is an essential subgroup of G. By $\tilde{\jmath}(G') \subseteq L$ and (2) we get $\tilde{\jmath}(G') \cap T_1 = \{0\}$. Since $\tilde{\jmath}(G_1)$ is an essential subgroup of G_1 this yields $G_1 \cap T_1 = \{0\}$.

To finish the proof assume that (b_2) holds. Then the hypotheses of Lemma 1.14 are satisfied. This gives the desired embedding $\mathbf{P}(\sigma) \hookrightarrow \mathscr{P}_\sigma(G)$. □

3.3. Theorem. *Let $\mathfrak{B}$ be a variety of Abelian groups, let $\sigma > \omega$ and let $G \in \mathfrak{B}$ with $|G| \leq 2^\sigma$. If G has a $\mathfrak{B}$-free subgroup F with* $\mathrm{Ps}(|F|, \sigma)$, *then $\mathbf{P}(\sigma) \hookrightarrow \mathscr{P}_\sigma(G)$. If $\mathfrak{B} = \mathfrak{A}$, then $\mathbf{P}(\sigma) \hookrightarrow CLC\mathscr{P}_\sigma(G)$.*

Proof. Note first that the injective compact generator H as in (a) of Proposition 3.2 can be chosen in such a way to satisfy also $\mathfrak{A} = \mathrm{qvar(H)}$. For $\mathfrak{B} = \mathfrak{A}$ and $H = \mathbb{T}$ and $H = \mathbb{Z}(n)$ for $\mathfrak{B} = \mathfrak{A}_n$ it follows from the simple fact that $\mathfrak{B} = \mathrm{var}(H) = \mathrm{qvar}(H)$. Now we can apply Proposition 3.2 to our group G to get the desired embedding $\mathbf{P}(\sigma) \hookrightarrow \mathscr{P}_\sigma(G)$. For $\mathfrak{B} = \mathfrak{A}$ it suffices to observe that $H = \mathbb{T}$ is connected and locally connected. It follows from Lemma 1.16 that also $\mathbb{T}^\sigma$ has the same properties. Moreover, both properties are preserved by taking dense pseudocompact subgroups and topological direct summands according to Lemma 1.16. At this point we are in position to apply Lemma 1.15 to the group G, the compact group $\mathbb{T}^\sigma$, the family $\mathscr{N}$ of subgroups of $\mathbb{T}^\sigma$ produced in the proof of Proposition 3.2 and the property "connected and locally connected". This will produce a family of connected and locally connected topologies $\mathscr{T}$ in $\mathscr{P}_\sigma(G)$ anti-isomorphic to $\mathscr{N}$. By property (iv) of Lemma 2.2 the family $\mathscr{N}$ is isomorphic to $\mathbf{P}(\sigma)$, then also the family $\mathscr{T}$ will be isomorphic to $\mathbf{P}(\sigma)$. □

3.4. Corollary. *Let $G \in \mathfrak{P}$ be an Abelian group with $r(G) = |G|$. Then $\mathbf{P}(\sigma) \hookrightarrow CLC\mathscr{P}_\sigma(G)$ for each uncountable cardinal σ with* $\mathrm{Ps}(|G|, \sigma)$. *In particular, G is a* CR-*group whenever $|G| > \mathfrak{c}$.*

Proof. Set $\tau = |G|$ and assume that $\mathrm{Ps}(\tau,\sigma)$ holds for some cardinal $\sigma > \omega$. This gives $\tau \leq 2^\sigma$. By our hypothesis we can find a free subgroup F of G with $|F| = \tau$. Then we can apply Theorem 3.3 with $\mathfrak{V} = \mathfrak{A}$, G and F to claim that $\mathbf{P}(\sigma) \hookrightarrow CLC\mathscr{P}_\sigma(G)$, in particular, $\mathscr{P}_\sigma(G) \neq \emptyset$. If $|G| > \mathfrak{c}$, then all σ with $\mathrm{Ps}(\tau,\sigma)$ must be uncountable. □

The above results is a purely Abelian phenomenon. Here we propose a comment for the non-Abelian case.

3.5. Remark. (a) To be able to transfer the argument of Proposition 3.2 and Theorem 3.3 in the non-Abelian case we need the following notion: for a variety $\mathfrak{V}$ a group $H \in \mathfrak{V}$ is a *cogenerator* of $\mathfrak{V}$ if every group of $\mathfrak{V}$ is a subgroup of some power of H. Clearly, H is a cogenerator of $\mathfrak{V}$ iff $\mathfrak{V}$ coincides with the quasi-variety $\mathrm{qvar}(H)$ generated by H (or equivalently, for every $G \in \mathfrak{V}$ and $g \in G$, $g \neq 1$, there should exist a homomorphisms $f : G \to H$ with $f(g) \neq 1$). Clearly a $\mathfrak{V}$-injective generator H of $\mathfrak{V}$ is also a cogenerator if H contains a copy of any cyclic simple group in $\mathfrak{V}$. This occurs when $\mathfrak{V} \subseteq \mathfrak{B}_n$ (see Theorem 3.6 and Claim 3.7).

(b) Every group of a variety $\mathfrak{V}$ with compact cogenerator is maximally almost periodic. Conversely, if all groups of a variety $\mathfrak{V}$ are maximally almost periodic, then $\mathfrak{V}$ has a compact cogenerator. In fact, $\mathfrak{V}$ is obviously cogenerated by its compact groups. Since $\mathfrak{V} \neq \mathfrak{A}$ (see item (c)), the non-Abelian compact Lie groups of $\mathfrak{V}$ are finite. Hence $K_{\mathfrak{V}} \times K$ is a compact cogenerator of $\mathfrak{V}$, where K is a compact cogenerator of $\mathfrak{V} \cap \mathfrak{A}$, i.e. $K = \mathbb{T}$ if $\mathfrak{A} \subseteq \mathfrak{V}$ and $K = \mathbb{Z}(n)$ if $\mathfrak{V} \cap \mathfrak{A} = \mathfrak{A}_n$.

(c) By (b) a variety $\mathfrak{V}$ with compact cogenerator is precompact. There is a lot of precompact varieties $\mathfrak{V}$ which do not have a compact cogenerator, i.e. contain a non-maximally almost periodic group. For $\mathfrak{G}$ one can take the group $\mathrm{SL}_2(\mathbb{C})$ which admit no nontrivial homomorphism into a compact group. Actually, every variety containing the variety $\mathfrak{N}_2$ of nilpotent groups of class 2 will do. In fact, there exists $G \in \mathfrak{N}_2$ such that the kernel R of the Bohr compactification $b_G : G \to G^\#$ of G is the commutator subgroup $G' = [G,G]$ of G (note that always $R \subseteq G'$ since the quotient G/G' is Abelian, thus maximally almost periodic). Two examples to this effect can be found in [44], Example 5.10 is a p-group of exponent p, where p may be any odd prime, while Example 5.11 is torsion-free.

(d) The variety $\mathfrak{G}$ admits no cogenerator at all. In fact, such a cogenerator should contain an isomorphic image of each simple group, but this is impossible since there are arbitrarily large simple groups (see also Theorem 3.6).

We have shown above (Proposition 3.2(a) and the proof of Theorem 3.3) that every Abelian variety $\mathfrak{V}$ has a $\mathfrak{V}$-injective compact cogenerator. The next theorem gives some restraints on varieties having this property. The following fact will be needed in the proof of the theorem:

Claim. *A $\mathfrak{V}$-injective generator H of a variety $\mathfrak{V}$ is also a cogenerator of $\mathfrak{V}$ iff $\mathfrak{V} \cap \mathfrak{A} \subseteq qvar(H)$, i.e. H contains a copy of any cyclic simple group in $\mathfrak{V}$.*

Proof. Follows directly from the definitions. □

3.6. Theorem. *Let $\mathfrak{V}$ be a variety which has a $\mathfrak{V}$-injective compact generator H. Then*:

(a) *either $\mathfrak{V} = \mathfrak{A}$ or $\mathfrak{A}_n \subseteq \mathfrak{V} \subseteq \mathfrak{B}_n^f$ for some n*;

(b) *if $\mathfrak{A}_n \subseteq \mathfrak{V} \subseteq \mathfrak{B}_n^f$ then H is also a cogenerator of $\mathfrak{V}$*;

(c) *for every simple non-Abelian group $S \in \mathfrak{V}$ all Abelian subgroups of S are cyclic of prime order* (*in particular, $\mathfrak{V}$ contains no finite simple non-Abelian groups*).

Proof. (a) Let H be a $\mathfrak{V}$-injective compact generator of $\mathfrak{V}$. Assume $\mathbb{Z} \in \mathfrak{V}$, i.e. $\mathfrak{A} \subseteq \mathfrak{V}$. Then H is divisible, hence connected. Therefore either $\mathfrak{V} = \mathfrak{A}$, or $\mathfrak{V} = \mathfrak{G}$ by [23, Theorem 3.25]. Let us see that the second case cannot occur. In fact, if H were non-Abelian, then the connected group H would contain a compact subgroup L which is a simple Lie group. In particular, L has few automorphisms. On the other hand, L contains a free subgroup F isomorphic to $F_{\mathfrak{c}}$ (cf. [1]) which has $2^{\mathfrak{c}}$ automorphisms. Hence some of them cannot be extended to homomorphisms $L \to H$. This proves that $\mathfrak{V} \neq \mathfrak{G}$. (An alternative proof is possible by showing that $\mathfrak{G}$ has no $\mathfrak{G}$-injective groups.) Therefore we have $\mathfrak{V} = \mathfrak{A}$ in this case.

Now assume that $\mathbb{Z} \notin \mathfrak{V}$. Then $\mathfrak{V} \subseteq \mathfrak{B}_n$ for some n. Since $\mathfrak{V}$ is generated by its compact groups, we have actually $\mathfrak{V} \subseteq \mathfrak{B}_n^f$. Choose n to be minimal with this property. Then $\mathbb{Z}(n) \in \mathfrak{V}$, since otherwise there would be a proper divisor d of n such that $\mathfrak{V} \subseteq \mathfrak{B}_d$ and consequently $\mathfrak{V} \subseteq \mathfrak{B}_d^f$ for $\mathfrak{V}$ is generated by its compact groups – in contradiction with the choice of n. Hence $\mathbb{Z}(n) \in \mathfrak{V}$ and consequently, $\mathfrak{A}_n \subseteq \mathfrak{V}$.

(b) To show that H is a cogenerator of $\mathfrak{V}$ we make recourse to the above Claim, so that it suffices to show that H contains a copy of any cyclic simple group in $\mathfrak{V}$. In fact, suppose $C \in \mathfrak{V}$ is a cyclic group of order p. Then it suffices to see that p must divide the period of some element of H. Assume not. Then for $m = n/(n, p^n)$, where (n, p^n) is the GCD of n and p^n, all elements of H have period which divide m, i.e., $H \in \mathfrak{B}_m$. Then H cannot generate $\mathfrak{V}$ – a contradiction. Hence H contains a subgroup isomorphic to C.

(c) The case $\mathfrak{V} = \mathfrak{A}$ is trivial, hence we assume from now on that $\mathfrak{A}_n \subseteq \mathfrak{V} \subseteq \mathfrak{B}_n^f$. Now let $S \in \mathfrak{V}$ is a simple non-Abelian group and assume that A is a nonsimple Abelian subgroup of S. Then A (or an appropriate subgroup of A) admits a nontrivial quotient $f : A \to A_1$ with non-trivial $N = \ker f$ and A_1 cyclic of prime order. By (b) H is also a cogenerator of $\mathfrak{V}$, thus there exists a nontrivial homomorphism $\varphi : A_1 \to H$. Now the composition $\varphi \circ f : A \to H$ admits an extension $\overline{f} : S \to H$. Then $\ker \overline{f}$ will be a proper normal subgroup of S by the equality $\ker \overline{f} \cap A = N \neq 1$ – a contradiction. To finish the proof of (c) note that a finite simple non-Abelian group always contains Abelian subgroups which are not simple. □

3.7. Remark. (a) The above theorem restricts strongly the varieties $\mathfrak{V}$ which have a $\mathfrak{V}$-injective compact generator. For example, none of the varieties $\mathfrak{B}_{60n}^f$ has a $\mathfrak{B}_{60n}^f$-

injective generator (note that $\mathfrak{B}_n^f$ contains a finite group S iff $|S|/n$). Here, of course, $60 = |A_5|$ can be replaced by any other cardinality of a finite simple non-Abelian group. This examples leave open the question if $\mathfrak{B}_{p^n}^f$ may have a $\mathfrak{B}_{p^n}^f$-injective generator, since now all groups of $\mathfrak{B}_{p^n}^f$ are nilpotent. We do not know even whether all groups of this variety are maximally almost periodic.

(b) We cannot claim in (b) of Theorem 3.6 that H is a cogenerator of $\mathfrak{B}$ also when $\mathfrak{B} = \mathfrak{A}$, in fact now any compact divisible torsion-free group H may serve as a counter-example. Anyway, the variety $\mathfrak{A}$ has a compact $\mathfrak{A}$-injective cogenerator, namely $\mathbb{T}$.

(c) Item (c) of Theorem 3.6 does not eliminate *all* simple groups in $\mathfrak{B}$. In fact, there exist infinite simple groups in which all proper subgroups are cyclic of a fixed order p (the so called Tarski monsters).

3.8. Question. (a) Let $\mathfrak{B}$ be a precompact variety which has a $\mathfrak{B}$-injective cogenerator. Must $\mathfrak{B}$ have also a $\mathfrak{B}$-injective compact generator (or, at least, a compact cogenerator)?

(b) Let $\mathfrak{B}$ be a variety which has compact cogenerator. Must $\mathfrak{B}$ have also a $\mathfrak{B}$-injective compact generator?

(c) Let $\mathfrak{B}$ be a variety which has $\mathfrak{B}$-injective compact generator. Is $\mathfrak{B}$ necessarily contained in $\mathfrak{A}$?

A negative answer to (c) will justify the following:

3.9. Problem. Find appropriate versions of 3.2–3.4 for varieties $\mathfrak{B}$ with $\mathfrak{B}$-injective compact generator.

4. Chains of pseudocompact topologies on torsion Abelian groups

Since pseudocompact torsion Abelian groups are bounded by [14, 7.4], in this section we restrict ourselves to considering only bounded torsion groups.

Let G be a bounded torsion Abelian group.

(i) For each $p \in \mathbb{P}$ we let $G_p = \{g \in G: p^k g = 0 \text{ for some } k \in \mathbb{N}\}$.

(ii) Set $\beta_{k,p}^G = r_p(G_p/G_p[p^k]) = r_p(p^k G_p)$ for $k \in \mathbb{N}$ and $p \in \mathbb{P}$. Note that in case this cardinal is infinite, we have $\beta_{k,p}^G = |G_p/G_p[p^k]| = |p^k G_p|$.

(iii) Since G is a direct sum of cyclic groups by Prüfer's theorem [27, Theorem 17.2], for each $p \in \mathbb{P}$ either $G_p = \{0\}$, or there exist an integer $r_p^G \geq 1$ and cardinals $\alpha_{0,p}^G, \ldots, \alpha_{r_p^G,p}^G$ such that $\alpha_{r_p^G,p}^G > 0$ and $G_p = \oplus\{\mathbb{Z}(p^{k+1})^{(\alpha_{k,p}^G)} : 0 \leq k \leq r_p^G\}$. More precisely, one can see that $\alpha_{k,p}^G = r_p(p^k G[p]/p^{k+1}G[p])$ – known as *Ulm–Kaplansky invariant* of G [27, p. 154]. Obviously, $\beta_{k,p}^G = \sum_{i=k}^r \alpha_{i,p}^G$ for all $k \in \mathbb{N}$ and $p \in \mathbb{P}$.

It is well known that a bounded torsion Abelian group G admits a compact group topology iff all Ulm–Kaplansky invariants of G are either finite or exponential [29, Section 25].

According to Theorem 6.2 of [23] a bounded torsion Abelian group G has a pseudocompact group topology iff every cardinal $\beta^G_{k,p}$ is either finite or admissible.

4.1. Theorem. *For a bounded torsion Abelian group G and a cardinal $\sigma > \omega$ the following are equivalent*:

(a) $\mathscr{P}_\sigma(G)$ *contains a copy of the Boolean algebra* $\mathbf{P}(\sigma)$;

(b) G *has a pseudocompact group topology of weight* σ;

(c) G *has a pseudocompact group topology and* $\mathrm{Ps}(|G|,\sigma)$ *holds*;

(d) $\mathrm{Ps}(|G|,\sigma)$ *holds and for all* $m \in \mathbb{N}$, $|mG| = |G/G[m]|$ *is either finite or admissible*;

(e) $\mathrm{Ps}(|G|,\sigma)$ *holds and every cardinal* $\beta^G_{k,p}$ *is either finite or admissible.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. The implication (d) $\Rightarrow$ (e) can be easily shown as in the proof of [23, Theorem 6.2] (see also [20, Theorem 5.2]). To prove the implication (c) $\Rightarrow$ (d) note that the multiplication by m is a continuous surjective homomorphism $G \to mG$, hence $mG \in \mathfrak{P}$ by the properties of pseudocompactness (see [23, Theorem 6.2], or [20, Theorem 5.2]).

We prove the remaining implication (e) $\Rightarrow$ (a) for every cardinal σ. We need the following definition only for this proof: for $p \in \mathbb{P}$ we call an infinite bounded p-group G *tame* if the leading Ulm–Kaplansky invariant of G coincides with the cardinality of G, i.e. $G = \bigoplus\{\mathbb{Z}(p^k)^{(\alpha_k)} : 1 \le k \le r\}$, where $\alpha_1,\dots,\alpha_r$, $r \ge 1$, is a finite sequence of cardinals and $\alpha_k \le \alpha_r$ for all $k = 1,\dots,r$. In terms of the cardinals $\beta^G_{k,p}$ defined in (ii) above G is tame iff all $\beta^G_{k,p}$ are equal to $|G|$.

Next we note that for a tame p-group G the implication (e) $\Rightarrow$ (a) of Theorem 4.1 holds. In fact, now the group G satisfies the hypothesis of Theorem 3.3 with $\mathfrak{V} = \mathfrak{A}_{p^n}$ where p^n is the exponent of G and $F = \mathbb{Z}(p^n)^{(|G|)}$.

Now we pass to the general case of a p-group. Assume that every cardinal $\beta^G_{k,p}$ is either finite or admissible. We show then that for each prime p the poset $\mathscr{P}_\sigma(G_p)$ contains a copy of the Boolean algebra $\mathbf{P}(\sigma)$ for each σ satisfying $\mathrm{Ps}(|G_p|,\sigma)$.

Consider first the case of a p-group $G = G_p$. If G is tame then there is nothing to prove. We proceed by induction on the number of nonzero Ulm–Kaplansky invariants of G. If this number is 1, then $G = \mathbb{Z}(p^{r+1})^{(\alpha_r)}$ is an $\mathfrak{A}_{p^{r+1}}$-free group, so in particular tame. Suppose G is not tame and that our claim was already proved for all groups satisfying (e) and having less Ulm–Kaplansky invariants than G. Let $t = \max\{i < r : \alpha_i = |G|\}$. Then $t < r$ and $G = G' \oplus G''$ where $G' = \bigoplus\{\mathbb{Z}(p^{k+1})^{(\alpha_k)} : 1 \le k \le t\}$ is tame with $|G'| = |G|$. Consider the group G'', having less nonzero Ulm–Kaplansky invariants than G. From the choice of t and our assumption it follows that the cardinal $\beta^{G''}_{k,p} = \alpha_k + \cdots + \alpha_r = \beta^G_{k,p}$ is either finite or admissible for every $k > t$. So we can apply the inductive hypothesis to the group G''. In particular, we can claim that $G'' \in \mathfrak{P}$.

Hence the proof of this case can be concluded by the following

4.2. Claim. *Let* $G = \bigoplus_{i=1}^{n} G_i$ *be an infinite group with* $|G| = |G_1|$. *Assume that* G_i *is a* CR-*group for* $2 \leq i \leq n$. *Then*:

(a) *if* G_1 *is a* CR-*group, then also* G *is a* CR-*group*;

(b) *if for each cardinal* σ $\mathrm{Ps}(|G_1|,\sigma)$ *implies that* $\mathscr{P}_\sigma(G_1)$ *contains a copy of* $\mathbf{P}(\sigma)$, *then also* G *has the same property.*

Proof. In both cases it suffices to consider the case $n = 2$ and then apply induction. Let $\mathrm{Ps}(|G|,\sigma)$ hold for some σ. Then $\mathrm{Ps}(|G_1|,\sigma)$ holds. By $G_2 \in \mathfrak{P}$ there exists σ' such that $\mathrm{Ps}(|G_2|,\sigma')$ holds. Then by Lemma 1.2 (c) also $\mathrm{Ps}(|G_2|,\min\{\sigma,\sigma'\})$ holds.

(a) Assume G_1 is a CR-group. Then $\mathrm{Ps}(|G_1|,\sigma)$ yields that there exists a pseudocompact group topology T_1 on G_1 of weight σ and a pseudocompact group topology T_2 on G_2 of weight $\min\{\sigma,\sigma'\} \leq \sigma$. Then the product topology on G is pseudocompact and has weight σ.

(b) Choose a family $\mathscr{T} = \{T_\alpha\}$ of pseudocompact group topologies on G_1 of weight σ order-isomorphic to $\mathbf{P}(\sigma)$. Since G_2 is a CR-group there exists a pseudocompact group topology T_2 on G_2 of weight $\min\{\sigma,\sigma'\} \leq \sigma$. Then the family $\tilde{\mathscr{T}} = \{T_\alpha \times T_2\}$ is contained in $\mathscr{P}_\sigma(G)$ and order-isomorphic to $\mathbf{P}(\sigma)$. □

Now we finish the proof of Theorem 4.1 in the general case. Assume that every cardinal $\beta_{k,p}^{G}$ is either finite or admissible. We have shown above that for each prime p the poset $\mathscr{P}_\sigma(G_p)$ contains a copy of the Boolean algebra $\mathbf{P}(\sigma)$ for each σ satisfying $\mathrm{Ps}(|G_p|,\sigma)$. In particular, each G_p is a CR-group. We now apply item (b) of Claim 4.2 with G_1 the p-primary component G_p satisfying $|G| = |G_p|$. □

Now we can answer positively Question 0.1 (and in particular, [12, Question 3.21]) for torsion Abelian groups by showing that these groups are CR-groups. We show in the next section that the answer to Question 0.1 is negative in the case of divisible Abelian groups.

4.3. Corollary. *Torsion Abelian groups in* $\mathfrak{P}$ *of cardinality* $> \mathfrak{c}$ *are* CR-*groups.*

Proof. Follows from the equivalence of items (b) and (c) of Theorem 4.1. □

Our next two corollaries give some sufficient conditions under which a bounded torsion Abelian group admits long chains of pseudocompact group topologies.

4.4. Corollary. *Let* G *be a bounded torsion Abelian group having all Ulm–Kaplansky invariants either finite or admissible and let* $\sigma \geq \omega_1$. *Then for each infinite cardinal* λ *the following are equivalent*:

(a) *G has a bounded chain of size* λ *of pseudocompact group topologies of weight* σ.

(b) $\mathrm{Ps}(|G|,\sigma)$ *and* $C(\sigma,\lambda)$ *hold.*

In other words, for any torsion Abelian group $G \in \mathfrak{P}(\sigma)$ we get the maximum value $\mathrm{Ded}_{\mathrm{b}}(\mathscr{P}_\sigma(G)) = \mathrm{Ded}(\sigma)$.

4.5. Corollary. *Let G be an infinite bounded torsion Abelian group. Suppose that for every $p \in \mathbb{P}$ and any $k \leq r_p^G$ the Ulm–Kaplansky invariant $\alpha_{k,p}^G$ is either finite or $\sigma_{k,p}^{\omega} \leq \alpha_{k,p}^G \leq 2^{\sigma_{k,p}}$ for some infinite cardinal $\sigma_{k,p}$. Then $\mathscr{P}_\sigma(G)$ contains a copy of* $\mathbf{P}(\sigma)$ *for each cardinal σ satisfying* $\log|G| \leq \sigma \leq 2^{\max\{\sigma_{k,p}\}}$.

Next example shows that Theorem 4.1 is much stronger than both corollaries from it.

4.6. Example. (a) For an arbitrary $p \in \mathbb{P}$ set $G = \mathbb{Z}(p)^{(\omega)} \oplus \mathbb{Z}(p^2)^{(\mathfrak{c})}$. Since $\beta_{0,p}^G = \beta_{1,p}^G = \mathfrak{c}$ are $2^{\mathfrak{c}}$-admissible by Lemma 1.1, and all other $\beta_{k,q}^G$'s are equal to 0, G has a chain with top element of pseudocompact group topologies of weight $2^{\mathfrak{c}}$ and length $\max\{(2^{\mathfrak{c}})^+, 2^{(\mathfrak{c}^-)}\}$ (by Theorem 4.1, implication (c) $\Rightarrow$ (a), and the fact that both $C(2^{\mathfrak{c}}, (2^{\mathfrak{c}})^+)$ and $C(2^{\mathfrak{c}}, 2^{(\mathfrak{c}^+)})$ hold by Proposition 1.11; see [40] for a similar chain on the free Abelian group of rank $\mathfrak{c}$). However $\alpha_1 = \omega$ is not admissible, and there is no infinite cardinal $\sigma_{1,p}$ satisfying the inequalities from Corollary 4.5. This shows that both Corollary 4.4 and Corollary 4.5 are not applicable for G.

(b) Let F be a nontrivial finite Abelian group. Then by Corollary 4.4 $G = \bigoplus_\tau F \in \mathfrak{P}$ iff τ is admissible. In such a case G is a CR-group by Corollary 4.3 (see also [12, Theorem 3.3]). The example given in Remark 2.5(b) shows that the former fact cannot be extended to non-Abelian groups.

It is easy to see that if a group $G \in \mathfrak{P}$ with $|G| = \mathfrak{c}$ is a CR-group, then G necessarily admits a compact metrizable group topology. The next theorem shows that if G is also Abelian, this is also sufficient for being a CR-group.

4.7. Theorem. *Let G be an Abelian group with $|G| = \mathfrak{c}$. Then G is a* CR-*group iff it admits a compact metrizable group topology.*

Proof. As mentioned above, the necessity is valid also in the general case. Assume that G admits a compact metrizable group topology. Then either $|G| = r(G)$ $(= \mathfrak{c})$ or G is bounded torsion. In both cases $G \in \mathfrak{P}(\sigma)$ for every $\sigma > \omega$ such that $\mathrm{Ps}(\mathfrak{c}, \sigma)$ holds, i.e. for $\omega < \sigma \leq 2^{\mathfrak{c}}$ (Theorems 3.3 and 4.1). $\square$

As far as non-Abelian groups are concerned, we can mention that, as an easy corollary of [23, Corollary 5.14], the free group $F_{\mathfrak{c}}$ is not a CR-group, even if $F_{\mathfrak{c}} \in \mathfrak{P}(\sigma)$ for all $\omega < \sigma \leq 2^{\mathfrak{c}}$ (compare with Corollary 2.6).

5. Other Abelian groups: connected, divisible, torsion-free, etc

In this section we measure exclusively the subposet $C\mathscr{P}(G)$ of connected topologies in $\mathscr{P}(G)$ and the subposet $CLC\mathscr{P}(G)$ of locally connected group topologies of $C\mathscr{P}(G)$.

5.1. Theorem. *Let G be an Abelian group. Then for a cardinal $\sigma \geq \omega_1$ given the following conditions are equivalent*:

(i) $\mathrm{Ps}(r(G),\sigma)$ *and* $|G| \leq 2^\sigma$ *hold*;

(ii) *G admits a pseudocompact connected group topologies of weight σ*;

(iii) *there exists an embedding* $\mathbf{P}(\sigma) \hookrightarrow C\mathscr{P}_\sigma(G)$;

(iv) *there exists an embedding* $\mathbf{P}(\sigma) \hookrightarrow CLC\mathscr{P}_\sigma(G)$;

In the case $r_p(G) = 0$ for some prime p (in particular, when G is torsion-free) the above conditions are equivalent to the following one:

(v) *the poset of totally l-disconnected group topologies of $C\mathscr{P}_\sigma(G)$ contains a copy of the Boolean algebra $\mathbf{P}(\sigma)$.*

Proof. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii) are trivial. The implication (ii) $\Rightarrow$ (i) is [21, Theorem 3.14] (see also [23, Theorem 3.21] or [11, Theorem 4.15]).

To prove (i) $\Rightarrow$ (iv) we assume that $\mathrm{Ps}(r(G),\sigma)$ and $|G| \leq 2^\sigma$ hold for the cardinal σ. Let F be a free subgroup of G of cardinality $r(G)$. Then Theorem 3.3 applied to G, F, $\mathfrak{B} = \mathfrak{A}$ provides an embedding of $\mathbf{P}(\sigma)$ into $CLC\mathscr{P}_\sigma(G)$.

Now assume that $r_p(G) = 0$ hold for some prime p. To prove (i) $\Rightarrow$ (v) assume that $\mathrm{Ps}(r(G),\sigma)$ and $|G| \leq 2^\sigma$ hold. Then there exists a free subgroup $F \subseteq G$ of cardinality $r(G)$, so in particular $\mathrm{Ps}(|F|,\sigma)$ holds. We are going to apply Proposition 3.2 with $\mathfrak{B} = \mathfrak{A}$. To this end we need a compact divisible group H with $r_q(H) > 0$ for all primes $q \neq p$. One can take H to be the Pontryagin dual of the group $\mathbb{Q}_p$ of rationals with denominators p-powers. Then H is divisible since $\mathbb{Q}_p$ is torsion-free and $r_p(H) = 0$ since $\mathbb{Q}_p$ is p-divisible ([19, Chapter 3]), so that

$$r_p(H^\sigma) = 0 \tag{3}$$

as well. Since the group H is injective in $\mathfrak{A}$, it follows from Proposition 3.2 that there exists a dense embedding of G in H^σ as a dense pseudocompact subgroup and a family $\mathscr{N}$ of closed subgroup of H^σ satisfying conditions (i)–(iv) of Lemma 2.2. Obviously the group H^σ is connected, and we will see now that H^σ is totally l-disconnected. Assume K is a locally connected subgroup of H^σ. Then its closure $\overline{K}$ is a locally connected compact subgroup of H^σ. The connected component C of $\overline{K}$ is obviously open in $\overline{K}$. By [23, Theorem 8.5], applied to the connected locally connected compact group C, we can claim that the p-torsion part of C must be dense in C. By (3) it is trivial, hence $C = \{0\}$. This yields that K is finite. Thus H^σ is totally l-disconnected.

According to Lemma 1.16, the property of being connected and totally l-disconnected satisfies the hypothesis of Lemma 1.15, so we can get a family $\mathscr{T}$ of totally l-disconnected topologies in $C\mathscr{P}_\sigma(G)$ order-isomorphic to $\mathbf{P}(\sigma)$. □

5.2. Remark. (a) If G is Abelian and nonreduced, then G does not admit precompact zero-dimensional group topologies, i.e. $\mathscr{Z}\mathscr{B}(G) = \emptyset$. In this theorem one cannot add to the list of properties even "disconnected" since for a divisible group G one has $\mathscr{P}(G) = C\mathscr{P}(G)$ ([46, Theorem 2]).

(b) In case the group G satisfies $|G| = r(G)$ condition (i) reduces to only one condition, namely $\mathrm{Ps}(|G|, \sigma)$. This gives a new proof of Corollary 3.4.

(c) Even if an Abelian G admits a pseudocompact group topology of weight ω (i.e. compact metrizable topology), one cannot guarantee that they exist in huge quantity. In fact, for the group of p-adic integers $G = \mathbb{Z}_p$ we have $|\mathscr{P}_\omega(G)| = 1$. Here one can substitute the group $\mathbb{Z}_p$ with any product $G = \prod_{p \in \mathbb{P}}(\mathbb{Z}_p^{\sigma_p} \times \prod_{n=1}^{m_p} \mathbb{Z}(p^n)^{\alpha_{p,n}})$, where each σ_p, m_p and $\alpha_{p,n}$ is finite and these are the sole Abelian groups with this property [19, 18]. The class $\mathfrak{C}$ of reduced Abelian groups G admitting a unique, up to isomorphism, compact group topology was characterized implicitly by Orsatti [34] (for more details and further progress in this direction see [18]).

Now we consider the case of chains of connected pseudocompact group topologies on Abelian groups.

5.3. Theorem. *Let G be an Abelian group. Then for cardinals λ and $\sigma \geq \omega_1$ given the following conditions are equivalent:*

(i) $\mathrm{Ps}(r(G), \sigma)$, $|G| \leq 2^\sigma$ *and* $C(\sigma, \lambda)$ *hold;*

(ii) $CLC\mathscr{P}_\sigma(G)$ *admits a bounded chain of length* λ;

(iii) $C\mathscr{P}_\sigma(G)$ *admits a bounded chain of length* λ;

(iv) $\mathscr{B}_\sigma(G)$ *admits a bounded chain of length* λ *and* $C\mathscr{P}_\sigma((G)) \neq \emptyset$;

(v) $\mathscr{B}_\sigma(G)$ *admits a bounded chain of length* λ *and* $\mathrm{Ps}(r(G), \sigma)$ *hold.*

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (v) follows from Theorem 5.2. The implication (v) $\Rightarrow$ (i) follows from Fact 1.13 and $\mathscr{B}_\sigma(G) \neq \emptyset$.

(i) $\Rightarrow$ (ii) Assume $\mathrm{Ps}(\tau, \sigma)$, $|G| \leq 2^\sigma$ and $C(\sigma, \lambda)$ hold. Then by Proposition 5.2 the poset of connected and locally connected topologies in $\mathscr{P}_\sigma(G)$ contains a copy of $\mathbf{P}(\sigma)$. Now $C(\sigma, \lambda)$ applies. □

5.4. Corollary. *Let G be a torsion-free Abelian group and let λ be a cardinal. Then the following conditions are equivalent:*

(i) $CLC\mathscr{P}_\sigma(G)$ *admits a chain of length* λ *with both bottom and top elements;*

(ii) *the poset of totally l-disconnected group topologies of* $C\mathscr{P}_\sigma(G)$ *contains a copy of the Boolean algebra* $\mathbf{P}(\sigma)$ *and* $C(\sigma, \lambda)$ *holds;*

(iii) $\mathrm{Ps}(|G|, \sigma)$ *and* $C(\sigma, \lambda)$ *hold.*

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. The implication (iii) $\Rightarrow$ (ii) follows from Theorem 5.1 and the implication (i) $\Rightarrow$ (iii) follows from the above theorem. □

5.5. Corollary. *Let G be a divisible Abelian group. Then for cardinals λ and $\sigma \geq \omega_1$ given the following conditions are equivalent:*

(i) *$Ps(r(G),\sigma), |G| \leq 2^\sigma$ and $C(\sigma,\lambda)$ hold;*
(ii) *$CLC\mathscr{P}_\sigma(G)$ admits a chain of length λ with both bottom and top elements;*
(iii) *$\mathscr{P}_\sigma(G)$ admits a bounded chain of length λ;*
(iv) *$\mathscr{B}_\sigma(G)$ has a bounded chain of length λ, and $\mathscr{P}_\sigma(G) \neq \emptyset$.*

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. To prove (iv) $\Rightarrow$ (i) it suffices to note that every pseudocompact group topology on a divisible group is connected [46, Theorem 2], so that $\mathscr{P}_\sigma(G) \neq \emptyset$ yields $C\mathscr{P}_\sigma(G) \neq \emptyset$. By Theorem 5.1(i) yields $\mathbf{P}(\sigma) \hookrightarrow CLC\mathscr{P}_\sigma(G)$. This proves the implications (i) $\Rightarrow$ (ii). $\square$

The following corollary answers item (b) of Question 0.1 for divisible Abelian groups.

5.6. Corollary. *Every divisible Abelian group $G \in \mathfrak{P}$ with $|G| > \mathfrak{c}$ admits a chain of λ pseudocompact connected and locally connected group topologies of weight* $\log|G|$ *for every $\lambda < \mathrm{Ded}(\log|G|)$ (in particular, for $\lambda = (\log|G|)^+$). Consequently $G \in \mathfrak{P}(\log|G|)$.*

Proof. Since $G \in \mathfrak{P}$ there exists σ such that $G \in \mathfrak{P}(\sigma)$. For $\lambda = 1$ apply the above corollary to get $\mathrm{Ps}(r(G),\sigma)$ and $|G| \leq 2^\sigma$. Then $\log|G| \leq \sigma$, so that by Lemma 1.2(a3) this yields $\delta(\log|G|) \leq \delta(\sigma)$. Hence $\mathrm{Ps}(r(G), \log|G|)$ holds. Now take any $\lambda < \mathrm{Ded}(\log|G|)$. Since $\log|G| > \omega$, the above corollary implies that G admits a chain of length λ of pseudocompact connected and locally connected group topologies of weight σ. Observe that by Proposition 1.11 $C(\log|G|, \log|G|^+)$ holds, hence we can take $\lambda = (\log|G|)^+$. $\square$

The reader has surely observed that the above proof gives more. In fact, an Abelian group G admitting a connected pseudocompact group topology (in particular, an Abelian group with $|G| = r(G)$, or a torsion-free Abelian group) satisfies the conclusion of the above corollary, and hence $G \in \mathfrak{P}(\log|G|)$.

We show now that no divisible Abelian group $G \in \mathfrak{P}$ with $|G| = 2^{r(G)}$ (or, more generally, with $2^{r(G)} < 2^{|G|}$ and $|G|^\omega = |G|$) is a CR-group.

5.7. Theorem. *Let $G \in \mathfrak{P}$ be a divisible Abelian group. Then the following hold:*
(a) *$\Pi(G) \leq 2^{r(G)}$.*
(b) *If G is a* CR*-group and $|G|^\omega = |G|$, then $r(G) \geq \log 2^{|G|}$, i.e. $2^{r(G)} = 2^{|G|}$.*
(c) *If $|G| = 2^{r(G)}$, then $\Pi(|G|) \geq 2^{\Pi(G)}$. In particular, G is not a* CR*-group.*

Proof. (a) Suppose $G \in \mathfrak{P}(\sigma)$, then by Corollary 5.5 (applied with $\lambda = 1$) we conclude that $\mathrm{Ps}(r(G),\sigma)$ holds. By Lemma 1.2 this implies $r(G) \geq \log\sigma$ and consequently $\sigma \leq 2^{r(G)}$. Hence $\Pi(G) \leq 2^{r(G)}$.

(b) According to Lemma 1.5 $\mathrm{Ps}(|G|, 2^{|G|})$ holds, hence $\Pi(|G|) = 2^{|G|}$. Since G is a CR-group, $\Pi(G) = \Pi(|G|)$. Now (a) yields $2^{|G|} \leq 2^{r(G)}$. This proves (b) since obviously $|G| \geq r(G)$.

(c) Assume now $|G| = 2^{r(G)}$. By (a) $|G| = 2^{r(G)} \geq \Pi(G)$. Hence $\Pi(|G|) = 2^{|G|} \geq 2^{\Pi(G)}$. This yields $\Pi(|G|) \neq \Pi(G)$, hence G is not a CR-group. (Alternatively, if G were a CR-group (a) would give $|G| = 2^{r(G)} \geq 2^{|G|}$ – a contradiction.) □

We describe below some divisible Abelian CR-groups.

5.8. Theorem. *Let $G \in \mathfrak{P}$ be a divisible Abelian group such that $|G|$ is a strong limit cardinal. Then G is a* CR-*group.*

Proof. By Corollary 5.5 G admits pseudocompact group topologies only if $|G| = r(G)$. Consequently, according to Theorem 5.1, the group G admits pseudocompact group topologies of all possible weights (i.e. $G \in \mathfrak{P}(\sigma)$ holds for each σ satisfying $\mathrm{Ps}(|G|, \sigma)$). □

The above observations looks more impressive under GCH, in this case we can describe the divisible CR-groups (for further results see also [18]).

5.9. Corollary. *Assume GCH and let G be a divisible Abelian group G.*

(a) *If $|G|$ is a limit cardinal, then G admits pseudocompact group topologies iff $|G| = r(G)$.*

(b) *If $G \in \mathfrak{P}$, then the following are equivalent:*

(b_1) $|G| = r(G)$;

(b_2) *G is a* CR-*group.*

The following example shows that in general the answer to Question 0.1 is negative, i.e. there exist Abelian divisible groups in $\mathfrak{P}$ which are not CR-groups.

5.10. Example. Let G be a divisible Abelian group such that $r(G)^{\omega} = r(G)$ and $|G| = 2^{r(G)}$. Then according to Theorem 5.7 G is not a CR-group. On the other hand, by Lemma 1.1 $\mathrm{Ps}(r(G), 2^{r(G)})$ holds and obviously $|G| \leq 2^{2^{r(G)}}$, thus G admits a pseudocompact group topology of weight $\sigma = 2^{r(G)}$. In fact, $C\mathscr{P}_{\sigma}(G)$ contains a copy of $\mathbf{P}(\sigma)$ (Theorem 5.1).

In our next corollaries we substantially strengthen Theorems 3–5 of [9].

We give here explicitly the following corollary of Theorem 5.2.

5.11. Corollary. *If $\alpha = \alpha^{\omega}$, then for every Abelian group G with $\alpha \leq r(G) \leq 2^{\alpha}$ and for every cardinal σ satisfying* $\max\{\omega_1, \log|G|\} \leq \sigma \leq 2^{\alpha}$ *the poset $CLC\mathscr{P}_{\sigma}(G)$ of locally connected group topologies in $C\mathscr{P}_{\sigma}(G)$ contains a copy of $\mathbf{P}(\sigma)$, so the size of $CLC\mathscr{P}_{\sigma}(G)$ is at least 2^{σ} and $CLC\mathscr{P}_{\sigma}(G)$ contains a chain of length σ^{+}.*

Proof. Since α must obviously be infinite, an application of Lemma 1.1 suffices to conclude that $r(G)$ is σ-admissible. Since $|G| \leq 2^{\sigma}$, Theorem 5.1 applies. □

5.12. Corollary. *Let G be a torsion-free Abelian group which satisfies $\alpha = \alpha^{\omega} \leq |G| \leq 2^{\alpha}$ for some cardinal α. Then for every cardinal σ satisfying* $\max\{\omega_1, \log|G|\} \leq \sigma \leq 2^{\alpha}$ *the poset $CLC\mathscr{P}_{\sigma}(G)$ contains a copy of* $\mathbf{P}(\sigma)$, *so the size of $CLC\mathscr{P}_{\sigma}(G)$ is at least 2^{σ} and $CLC\mathscr{P}_{\sigma}(G)$ contains a chain of length σ^{+}. In particular, if G satisfies $2^{\beta} \leq |G| \leq 2^{2^{\beta}}$ for some cardinal β, then $CLC\mathscr{P}_{2^{\beta}}(G)$ contains a copy of* $\mathbf{P}(2^{\beta})$.

Proof. Theorem 5.1 applies in the first part. Now assume that G satisfies $2^{\beta} \leq |G| \leq 2^{2^{\beta}}$. Since G is infinite (being torsion-free), the cardinal β must be infinite. Then the first assertion of the corollary applies to the cardinal $\alpha = 2^{\beta}$. □

Let us note that in this corollary "torsion-free" can be replaced by "Abelian with $|G| = r(G)$". In case G is either free (i.e. $G = \bigoplus_{\gamma} \mathbb{Z}$) or divisible and torsion-free (i.e. $G = \bigoplus_{\gamma} \mathbb{Q}$) and $|G| < \sigma \leq 2^{\sigma}$ this corollary gives [11, Theorem 6.2], so in particular [9, Theorems 4 and 5] when G is a free group.

Now we give the counterpart regarding $\mathfrak{B}$-free groups. Since it remains true for other classes as well, we give it in the following more general form:

5.13. Corollary. *Let α be an infinite cardinal and G be a group having one of the following algebraic properties*: (i) *relatively free and residually finite*; (ii) *torsion Abelian*; (iii) *Abelian with $|G| = r(G)$.*

(a) *If $\alpha = \alpha^{\omega} \leq |G| \leq 2^{\alpha}$, then for every cardinal σ satisfying* $\max\{\omega_1, \log|G|\} \leq \sigma \leq 2^{\alpha}$ *the poset $\mathscr{P}_{\sigma}(G)$ contains a copy of* $\mathbf{P}(\sigma)$. *In particular, the size of $\mathscr{P}_{\sigma}(G)$ is at least 2^{σ} and $\mathscr{P}_{\sigma}(G)$ contains a chain of length σ^{+}.*

(b) *If $2^{\alpha} \leq |G| \leq 2^{2^{\alpha}}$, then for every cardinal σ such that* $\max\{\omega_1, \log|G|\} \leq \sigma \leq 2^{2^{\alpha}}$ *the poset $\mathscr{P}_{\sigma}(G)$ contains a copy of* $\mathbf{P}(\sigma)$, *so that the size of $\mathscr{P}_{\sigma}(G)$ is at least 2^{σ}. In particular, the size of $\mathscr{P}_{2^{2^{\alpha}}}(G)$ is at least $2^{2^{2^{\alpha}}}$ and $\mathscr{P}_{2^{2^{\alpha}}}(G)$ contains a chain of length* $\max\{(2^{2^{\alpha}})^{+}, 2^{(2^{\alpha})^{+}}\}$.

Proof. (a) An application of Lemma 1.5 suffices to conclude that $|G|$ is σ-admissible. Now Theorem 2.2 applies in case (i), Theorem 4.1 in case (ii) and Corollary 3.4 in case (iii).

(b) Follows from (a) applied with 2^{α} in place of α. □

Now take $G = \bigoplus_{\tau} \mathbb{Z}$ or $G = \bigoplus_{\tau} \mathbb{Q}$ in the above corollary. Then item (a) with $\sigma = 2^{\alpha}$ gives [11, Corollary 6.3(a)], while item (b) with $|G| < \sigma \leq 2^{2^{\alpha}}$ gives [11, Theorem 6.2(ii)], and with $\sigma = 2^{2^{\alpha}}$ gives [11, Corollary 6.3(b)].

6. Epilogue: unbounded chains

Here we discuss unbounded chains. Note that chains with cofinality $\leq \sigma$ in $\mathscr{B}_{\sigma}(G)$ are actually bounded, so that as far as *precompact* topologies are concerned, "bounded chains" can be replaced by "chains with cofinality $\leq \sigma$" in the sequel. The reader should note also that the the existence of a not necessarily bounded chain of length λ in $\mathscr{B}_\sigma(G)$ or $\mathscr{P}_\sigma(G)$ does not imply $C(\lambda,\sigma)$.

In terms of the cardinal function Ded we have proved in the preceding sections that $\mathrm{Ded_b}(\mathscr{P}_\sigma(G)) = \mathrm{Ded}(\sigma) = \mathrm{Ded_b}(\mathscr{B}_\sigma(G))$ for all groups considered in Sections 2–5. In the sequel (see Theorems 6.3 and 6.4) we will see that boundedness can often be relaxed in this equality, i.e. $\mathscr{P}_\sigma(G)$ has as long chains as $\mathscr{B}_\sigma(G)$, so that in terms of the function Ded

$$\mathrm{Ded}(\mathscr{P}_\sigma(G)) = \mathrm{Ded}(\mathscr{B}_\sigma(G)). \tag{3}$$

This justifies the study of $\mathrm{Ded}(\mathscr{B}_\sigma(G))$ which can be reduced to purely combinatorial computations depending only on the cardinal number $2^{|G|}$ in case G is Abelian or close to being Abelian as the next theorem shows. The condition $|G| < 2^{|G/G'|}$ is verified for relatively free groups (they satisfy a much stronger condition: $|G| = |G/G'|$).

6.1. Theorem. *Let G be an infinite group with $|G| < 2^{|G/G'|}$ and let σ be an infinite cardinal satisfying $\mathscr{B}_\sigma(G)\neq\emptyset$. Then there exists an embedding $\mathbf{P}_\sigma(2^{|G/G'|}) \hookrightarrow \mathscr{B}_\sigma(G)$. Consequently, if $2^{|G/G'|} = 2^{|G|}$ (in particular, if $|G/G'| = |G|$), then $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} \mathbf{P}_\sigma(2^{|G|})$, so that* $\mathrm{Ded}(\mathscr{B}_\sigma(G)) = \mathrm{Ded}(\mathbf{P}_\sigma(2^{|G|}))$.

A sharper form of this theorem was announced in [16], the proof appears in [3, Theorem 7.9]. In view of the properties of the function Ded given in Section 1.2, Theorem 6.1 implies that for an infinite group with $|G| < 2^{|G/G'|}$ and for an infinite cardinal σ with $\mathscr{B}_\sigma(G) \neq \emptyset$ the equality $\mathrm{Ded_b}(\mathscr{B}_\sigma(G)) = \mathrm{Ded}(\sigma)$ holds, in particular $\mathrm{Ded}(\mathscr{B}_{2^{|G|}}(G)) = \mathrm{Ded}(2^{|G|})$ provided $\mathscr{B}_{2^{|G|}}(G) \neq \emptyset$. Moreover, if $\sigma < 2^{|G|}$, then $\mathscr{B}_{2^{|G|}}(G) \neq \emptyset$ yields $\mathrm{Ded}(\mathscr{B}_\sigma(G)) = \mathrm{Ded}(\mathbf{P}_\sigma(\sigma^+))$, i.e. the existence of a chain of length λ in $\mathscr{B}_\sigma(G)$ yields the existence of such a chain in $\mathbf{P}_\sigma(\sigma^+)$, hence the cofinality of such a chain can be at most σ^+. Theorem 6.1 substantially strengthens the following result announced without proof in [11]: Let G be a group with $|G/G'| = |G|$ and suppose that a cardinal σ satisfies $\log|G| \leq \sigma \leq 2^{|G|}$. If $\mathscr{B}_\delta(G)\neq\emptyset$ for some $\delta \leq \sigma$ then the poset $\mathscr{B}_\sigma(G)$ contains a chain of length λ iff $\mathbf{P}_\sigma(2^{|G|})$ contains a chain of length λ. (Note that the last hypothesis yields $\mathscr{B}_\sigma(G)\neq\emptyset$ so that Theorem 6.1 applies.)

In view of the relations

$$\begin{aligned}\mathrm{Ded}(\sigma) &= \mathrm{Ded_b}(\mathscr{P}_\sigma(G)) \leq \mathrm{Ded}(\mathscr{P}_\sigma(G)) \leq \mathrm{Ded}(\mathscr{B}_\sigma(G))\\ &= \mathrm{Ded}(\mathbf{P}_\sigma(\sigma^+)) \leq \mathrm{Ded}(\sigma)^+\end{aligned} \tag{4}$$

it seems interesting to discuss also the question whether the unbounded chains of pseudocompact group topologies have the same lengths as the bounded ones, i.e. whether the equality

$$\mathrm{Ded_b}(\mathscr{P}_\sigma(G)) = \mathrm{Ded}(\mathscr{P}_\sigma(G)) \tag{5}$$

holds. Clearly the failure of either (3) or (5) implies $\mathrm{Ded}(\mathbf{P}_\sigma(\sigma^+)) = \mathrm{Ded}(\sigma)^+$ which by (iii) entails $\mathrm{cf}(\mathrm{Ded}(\sigma)) = \sigma^+$. Hence, by (4), at most one of the equalities

(3) and (5) may fail for a given G and σ. This proves the following curious fact:

6.2. Theorem. *Let G be a group having one of the following algebraic properties:* (i) *relatively free and residually finite*; (ii) *torsion Abelian*; (iii) *Abelian with* $|G| = r(G)$. *Then for every infinite cardinal σ with $\mathscr{P}_\sigma(G) \neq \emptyset$ either G has chains of pseudocompact group topologies of weight σ of the same length as those of the bounded ones, or G has chains of precompact group topologies of weight σ with the same length as those of the chains of pseudocompact ones.*

In other words, if the chains of pseudocompact group topologies of weight σ on G are "shorter" than those of the precompact ones, then bounded chains of pseudocompact group topologies of weight σ suffice to get the lengths of all chains of pseudocompact group topologies.

We offer the following theorem establishing simultaneously the above relations (3) and (5) under the condition $\sigma = 2^{|G|}$ or $2^\sigma = \sigma^+$.

6.3. Theorem. *Let G be an Abelian group and σ an infinite cardinal with $\mathscr{P}_\sigma(G) \neq \emptyset$. If either $2^\sigma = \sigma^+$ or $\sigma = 2^{|G|}$, then both* (3) *and* (5) *hold. In particular,* (3) *and* (5) *hold simultaneously under GCH.*

Proof. In general

$$\mathrm{Ded}_\mathrm{b}(\mathscr{B}_\sigma(G)) = \mathrm{Ded}_\mathrm{b}(\mathscr{P}_\sigma(G)) \leq \mathrm{Ded}(\mathscr{P}_\sigma(G)) \leq \mathrm{Ded}(\mathscr{B}_\sigma(G)),$$

hence it suffices to establish $\mathrm{Ded}_\mathrm{b}(\mathscr{B}_\sigma(G)) = \mathrm{Ded}(\mathscr{B}_\sigma(G))$ in both cases. The equality $2^\sigma = \sigma^+$ implies $\mathrm{Ded}_\mathrm{b}(\mathscr{B}_\sigma(G)) = \mathrm{Ded}(\sigma) = \sigma^{++} = \mathrm{Ded}(\mathscr{B}_\sigma(G))$. In the second case $\mathrm{Ded}_\mathrm{b}(\mathscr{B}_{2^{|G|}}(G)) = \mathrm{Ded}(\mathscr{B}_{2^{|G|}}(G))$ since now $\mathscr{B}_{2^{|G|}}(G)$ has a top element. □

In the following theorem we establish (3) under the condition $\mathrm{Ps}(|G|, \sigma^+)$ (beyond the necessary condition $\mathrm{Ps}(|G|, \sigma)$). The reader should note that this condition yields $\sigma < 2^{|G|}$ according to Lemma 1.1, but the case $\sigma = 2^{|G|}$ was already considered in the above theorem.

6.4. Theorem. *Let $G \in \mathfrak{P}$ be an infinite Abelian group which is either torsion or satisfies $r(G) = |G|$ (or G is torsion-free). Then the following are equivalent for every uncountable cardinal σ:*

(a) $\mathrm{Ps}(|G|, \sigma)$ *and* $\mathrm{Ps}(|G|, \sigma^+)$ *hold*;

(b) *there exists an embedding $\varphi : \mathbf{P}_\sigma(\sigma^+) \to \mathscr{P}_\sigma(G)$ such that $\varphi(\mathbf{P}_\sigma(\sigma^+))$ has an upper bound in $\mathscr{P}(G)$.*

Consequently, $\mathrm{Ded}(\mathscr{P}_\sigma(G)) = \mathrm{Ded}(\mathscr{B}_\sigma(G)) = \mathrm{Ded}(\mathbf{P}_\sigma(\sigma^+))$ *in the case* (a).

Proof. (a) ⇒ (b) The group G must be uncountable, so that the case $r(G) = |G|$ includes obviously the case when G is torsion-free. Assume that $r(G) = |G|$. Then there exists a subgroup N of G such that both N and G/N have the same properties as G, i.e. $|N| = r(N) = |G/N| = r(G/N) = r(G) = |G|$, $N \in \mathfrak{P}$ and $G/N \in \mathfrak{P}$.

Now assume that G is torsion and set $\alpha = |G|$. Our hypothesis $G \in \mathfrak{P}$ implies, in view of Theorem 4.1, that $G = G_1 \oplus G_2$, where $G_1 \in \mathfrak{P}$ is a p-group for some prime number p, $G_2 \in \mathfrak{P}$ has no nontrivial p-torsion elements and $|G_1| = \alpha \geq |G_2|$. Moreover, there exists $n \in \mathbb{N}^+$ such that $G_1 = C \oplus G_3$, where $C \cong \mathbb{Z}(p^n)^{(\alpha)}$ and $G_3 \cong G_1$. Since $G_2 \oplus G_3 \in \mathfrak{P}$ and $|G_2 \oplus G_3| = \alpha$, it follows from Theorem 4.1 and our hypothesis $\mathrm{Ps}(\alpha, \sigma)$ that $G_2 \oplus G_3$ admits a pseudocompact group topology T of weight σ. Now every chain $\{T_j : j \in J\}$ in $\mathscr{B}_\sigma(C)$ with an upper bound in $\mathscr{B}(C)$ will produce a chain $\{T_j \times T : j \in J\}$ in $\mathscr{B}_\sigma(G)$ of the same size with an upper bound in $\mathscr{B}(G)$. Therefore, we can assume without loss of generality that G is a p-group isomorphic to $\mathbb{Z}(p^n)^{(\alpha)}$ for some $n \in \mathbb{N}^+$. We note next, that in analogy to the case $r(G) = |G|$, we can find a subgroup N of G such that both N and G/N have the same properties as G, i.e. $|N| = |G/N| = \alpha$ and both N and G/N are isomorphic to $\mathbb{Z}(p^n)^{(\alpha)}$.

Let H denote a compact metrizable group, which is either $\mathbb{T}$ or $\mathbb{Z}(p^n)^\omega$ depending on whether G satisfies $|G| = r(G)$ or G is isomorphic to $\mathbb{Z}(p^n)^{(\alpha)}$. It follows from $\mathrm{Ps}(|N|, \sigma)$ and the proof of Theorem 3.2 that there exist a subset $C \subsetneq \sigma$ with $|C| = |\sigma \setminus C| = \sigma$ and a G_δ-dense monomorphism $i' : N \to H^\sigma$ such that $i'(N) \cap H^C = \{0\}$. Analogously we get a G_δ-dense monomorphism $j : G/N \to H^{\sigma^+}$. Further, arguing as in the proof in Theorem 3.2 we extend i' to a monomorphism $i : G \to H^\sigma$. Let $f : G \to H^{\sigma^+}$ be the composition of the canonical homomorphism $G \to G/N$ and j. This homomorphism need not be continuous, but by the choice of j the image $f(G)$ is a G_δ-dense subgroup of H^{σ^+}. Since $N \subseteq \ker f$ it follows that $\ker f$ is G_δ-dense subgroup of G equipped with the topology induced by the monomorphism i. Hence we can apply Claim 6.5 below to this data (see also [15]) to get that the subgroup $\Gamma_f = \{(i(x), f(x)) : x \in G\}$ of the group $L = H^\sigma \times H^{\sigma^+}$ is G_δ-dense in L, thus pseudocompact. Note that $\Gamma_f \cap H^{\sigma^+} = \{0\}$ since Γ_f is a graph. Let $\mathscr{N}$ be the family of closed subgroups of L defined as follows: for $A \in \mathbf{P}_\sigma(\sigma^+)$ set $N_A = H^{\sigma^+ \setminus A}$ considered as a subgroup of H^{σ^+}. Then $\mathscr{N}$ is anti-isomorphic to $\mathbf{P}_\sigma(\sigma^+)$ and for each $N_A \in \mathscr{N}$ we have $\Gamma_f \cap N_A = \{0\}$ and $w(L/N_A) = w(H^\sigma \times H^A) = \sigma$. By Lemma 1.15 applied to the group L, its subgroup $\Gamma_f \cong G$, the family $\mathscr{N}$ and the empty property $\mathscr{E}$ we conclude that $\mathscr{P}_\sigma(G)$ contains a copy of the poset $\mathbf{P}_\sigma(\sigma^+)$.

(b) $\Rightarrow$ (a) Assume that there exists an embedding $\varphi : \mathbf{P}_\sigma(\sigma^+) \to \mathscr{P}_\sigma(G)$ such that $\varphi(\mathbf{P}_\sigma(\sigma^+))$ has an upper bound in $\mathscr{P}(G)$. This gives $\mathscr{P}_\sigma(G) \neq \emptyset$ thus $\mathrm{Ps}(|G|, \sigma)$ holds. Let $T \in \mathscr{P}(G)$ be an upper bound for $\varphi(\mathbf{P}_\sigma(\sigma^+))$. Then $\sigma' = w(G, T) > \sigma$. In fact, since $\mathrm{height}(\mathbf{P}_\sigma(\sigma^+)) = \sigma^+$ we can find an ordinal chain of topologies $T_i \in \varphi(\mathbf{P}_\sigma(\sigma^+))$ of length σ^+. Then $T' = \sup\{T_i : i < \sigma^+\} \leq T$ and $w(G, T') = \sigma^+$. This proves $w(G, T) > \sigma$ since the weight is monotone for precompact group topologies (see item a) of Fact 1.13). Now $\mathscr{P}_{\sigma'}(G) \neq \emptyset$ yields $\mathrm{Ps}(|G|, \sigma')$. Since $\sigma < \sigma^+ \leq \sigma'$ this gives $\mathrm{Ps}(|G|, \sigma^+)$.

The last assertion is obvious. □

6.5. Claim. *Let G be a G_δ-dense subgroup of a topological group K and let $i : G \to K$ denote the inclusion. Assume $f : G \to K'$ is a (not necessarily continuous)* *homomorphism into a topological group K' such that $f(G)$ is a G_δ-dense subgroup of K' and* $\ker f$ *is G_δ-dense in G. Then the graph $\Gamma_f = \{(i(x), f(x)) : x \in G\}$ of f is a G_δ-dense subgroup of $K \times K'$.*

Proof. Let $U \subseteq K$, $V \subseteq K'$ be nonempty G_δ-dense subsets of K and K'. Then there exists $x \in G$ such that $f(x) \in V$. Consider the inverse image $A = \{y \in G : f(y) = f(x)\}$. This is a coset of $\ker f$ thus G_δ-dense subset of G by hypothesis and homogeneity of G. Since $U_1 = U \cap G = i^{-1}(U)$ is a nonempty G_δ-dense subset of G there exists $y \in U_1 \cap A$. Thus $f(y) = f(x) \in V$, while $i(y) \in U$. Therefore $(i(y), f(y)) \in U \times V$. □

We do not know if Theorem 6.4 can be extended to non-Abelian groups as well (say, relatively free residually finite groups, or even more general, to groups G with $|G/G'| = |G|$).

6.6. Corollary. *Let $G \in \mathfrak{P}$ be an infinite Abelian group which is either torsion or satisfies $r(G) = |G|$. If* $\mathrm{Ps}(|G|, 2^{|G|})$ *holds, then* $\mathrm{Ded}(\mathscr{P}_\sigma(G)) = \mathrm{Ded}(\mathscr{B}_\sigma(G))$ *for all* $\sigma \in [\log|G|, 2^{|G|}]$.

According to Lemma 1.5 and 1.6, to violate $\mathrm{Ps}(|G|, 2^{|G|})$ one needs $|G| \neq |G|^\omega$, moreover, if $\mathrm{cf}(|G|) = \omega$ (which is equivalent to $|G| \neq |G|^\omega$ under SCH), then $2^{<|G|} < 2^{|G|}$ must hold.

We note that the existence of an embedding as in item (b) of Theorem 6.4 is a sufficient condition for (3), but we do not know if it is also necessary in general.

6.7. Question. Is the equality $\mathrm{Ded}(\mathscr{P}_\sigma(G)) = \mathrm{Ded}(\mathscr{B}_\sigma(G))$ true whenever $\mathscr{P}_\sigma(G) \neq \emptyset$?

6.8. Remark. Here we offer a brief discussion about the possibility to (dis)prove (3) in SCH or (M). As already noted above, the failure of (3) implies $\mathrm{cf}(\mathrm{Ded}(\sigma)) = \sigma^+$. Further, in order to disprove (3) one has to ensure the failure of the hypotheses of Theorem 6.4 too. This means to find cardinals σ and τ satisfying $\omega < \sigma < 2^\tau$ and such that condition $\mathrm{Ps}(\tau, \sigma^+)$ fails while conditions $\mathrm{Ps}(\tau, \sigma)$ and $\mathrm{cf}(\mathrm{Ded}(\sigma)) = \sigma^+$ hold. Assume (M). Then by Proposition 1.9 the failure of $\mathrm{Ps}(\tau, \sigma^+)$ gives $\sigma^+ \geq 2^{<\sqrt{\tau}}$. But $\sigma^+ = 2^{<\sqrt{\tau}}$ is impossible by the second part of that proposition in view of the failure of $\mathrm{Ps}(\tau, \sigma^+)$ again. Now $\sigma^+ > 2^{<\sqrt{\tau}}$ and $\Pi(\tau) = 2^{<\sqrt{\tau}}$ imply that $\sigma = 2^{<\sqrt{\tau}}$. The second part of Proposition 1.9 and $\mathrm{Ps}(\tau, \sigma)$ give $\log 2^{<\sqrt{\tau}} < \sqrt{\tau}$, i.e. $2^{<\sqrt{\tau}}$ is not a proper limit. Therefore, to violate (3) under SCH one has to take $\omega = \mathrm{cf}(\tau) < \mathrm{cf}(2^{<\tau})$ and $\sigma = 2^{<\tau}$. Moreover, the cofinality of $\mathrm{Ded}(\sigma)$ must be σ^+. Then any Abelian CR-group of cardinality τ will work (say, a torsion-free group). We are not aware if such a choice of τ is possible in some appropriate model of ZFC satisfying SCH, in any case the example given below in the proof of Theorem 6.9 with $\mathrm{Ded}(\omega_1) < \mathrm{Ded}(\mathbf{P}_{\omega_1}(\omega_2))$ cannot help in this case since then (4) fails (so that (3) holds). (The smallest possible counterexample could exist for $\tau = \omega_\omega > \mathfrak{c}$ and $\sigma = \tau^+$. We do not know if one can also arrange to have $\mathrm{cf}(\mathrm{Ded}(\tau^+)) = \tau^{++}$.)

Now we show that the equality (5) cannot be proved in ZFC.

6.9. Theorem. *The equality*

$$\mathrm{Ded}(\mathscr{P}_{\omega_1}(\mathbb{R})) = \mathrm{Ded}_\mathrm{b}(\mathscr{P}_{\omega_1}(\mathbb{R})) \tag{6}$$

cannot be decided in ZFC.

Proof. Under GCH, (6) holds as proved in the above Theorem 6.4. It is shown in [3] that there exists a model $\mathfrak{M}$ of ZFC in which $\mathrm{Ded}(\mathbf{P}_{\omega_1}(\omega_2)) > \mathrm{Ded}(\omega_1)$. Note that both $\mathrm{Ps}(\mathfrak{c}, \omega_1)$ and $\mathrm{Ps}(\mathfrak{c}, \omega_2)$ hold. Thus for the group $\mathbb{R}$ (and for any torsion-free Abelian group of cardinality $\mathfrak{c}$) we have $\mathrm{Ded}(\mathscr{P}_{\omega_1}(\mathbb{R})) = \mathrm{Ded}(\mathbf{P}_{\omega_1}(\omega_2))$ by Theorem 6.4, while $\mathrm{Ded}_\mathrm{b}(\mathscr{P}_{\omega_1}(\mathbb{R})) = \mathrm{Ded}(\omega_1) < \mathrm{Ded}(\mathscr{P}_{\omega_1}(\mathbb{R}))$. Therefore (6) fails in this case. □

Let us note that in Mitchell's [31] model $\mathfrak{M}$ of ZFC applied in [3] one has $\mathrm{Ded}(\mathbf{P}_{\omega_1}(\omega_2)) = \mathrm{Ded}(\omega_1)^+$ and $\mathrm{Ded}(\omega_1) = 2^{\omega_1}$. Hence, in this model $\mathbb{R}$ has unbounded chains of pseudocompact group topologies of weight ω_1 of length 2^{ω_1}, while every bounded chain of such topologies has length $< 2^{\omega_1}$.

Historical remark. The first results of this paper (namely: a preliminary version of the key Lemma 2.2, the equivalence of (iii), (iv) and (v) in Corollary 2.4, Theorem 5.3, the equivalence of (i) and (iii) in Corollaries 5.4 and 5.5) were obtained in October 1990 jointly with Dmitri Shakhmatov after having seen an unpublished version of [8] in September 1990. Later the author developed the approach based on embedding of large posets in $\mathscr{P}_\sigma(G)$ and $\mathscr{B}_\sigma(G)$ and obtained the results on bounded chains (Sections 2–5) by the end of 1992. The question of whether unbounded chains (in $\mathscr{P}_\sigma(G)$ or $\mathscr{B}_\sigma(G)$) can be "longer" than the bounded ones was the starting point of [3] in December 1992. In 1993, the author finished Section 6, helped by the knowledge of some of the main results of [3]. The main results of this paper were announced at the Conference on Groups, Galway, Ireland, 1993 [16]. Chains and other features of the poset of *precompact* group topologies were discussed also in a survey-talk at the Colloquium on Topology, Szekszàrd, Hungary, 1993 [17].

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